DIMENSIONAL ANALYSIS

INTRODUCTION

Dimensional Analysis is a mathematical technique that makes use of the dimensions as a tool to the solution of several engineering problems. Each physical phenomenon can be expressed by an equation composed of physical quantities (or variables). These physical quantities may be dimensional or non-dimensional quantities. Through dimensional analysis, the physical quantities or variables can be arranged in a systematic fashion and the physical quantities can be combined to form non-dimensional parameters.

Uses of dimensional analysis in the study of fluid mechanics:

1. Testing the dimensional homogeneity of any equation in fluid mechanics
2. Deriving equations expressed in terms of non-dimensional parameters to show the relative significance of each parameter
3. Planning model tests and presenting experimental results in a systematic manner using non-dimensional parameters; this enables analysis of even complex fluid flow phenomenon.

DIMENSIONS

Engineers and scientists use various physical quantities to describe a physical phenomenon. These physical quantities can be described by a set of quantities which are in a sense independent of each other. These quantities are called ***fundamental quantities*** or ***primary quantities***.

The primary quantities are mass, length, time, and temperature denoted by M, L, T and θ respectively.

All other physical quantities such as area, volume, acceleration, force, energy, power, etc. are termed as ***derived quantities*** or ***secondary quantities***. These quantities are called secondary quantities because they can be expressed in terms of physical quantities.

The expression for a derived quantity in terms of the primary quantities is called the ***dimension*** of the physical quantity. For instance, let us derive the dimension of the derived quantity namely, force.

As per Newton's second law of motion, the dynamic force is the product of mass and acceleration. Acceleration, too, is a derived quantity which is the rate of change of velocity. Velocity is yet another derived quantity which represents the rate of change of displacement. The dimensions of velocity are: LT^{-1}. Hence, the dimensions of acceleration are: LT^{-2}; so, the dimensions of force are: MLT^{-2}.

Some engineers prefer to use force instead of mass as fundamental quantity because force is easy to measure. In such a case, the physical phenomenon is represented by variables expressed in F-L-T system instead of M-L-T system. The advantage with the dimensional form of any quantity is that it is independent of the system of units and enables us to convert from one system of units to the other system of units.

DIMENSIONAL HOMOGENEITY

The Fourier's principle of dimensional homogeneity states that an equation which expresses a physical phenomenon must be algebraically correct and dimensionally homogeneous.

When an equation is said to be dimensionally homogeneous?

An equation is said to dimensionally homogeneous, if the dimensions of the terms on the left hand side of the equation are same as the dimensions of the terms on the right hand side of the equation.

Illustration of dimensional homogeneity

Consider the expression for discharge in a rectangular weir,

$Q = (2/3)C_d(2g)^{1/2} LH^{3/2}$

Let us list the SI units and dimensions of the various quantities in the above expression

Quantity	SI units	Dimensions (M-L-T system)
Discharge, Q	m^3/s	L^3T^{-1}
Coefficint of discharge, C_d	No units	Dimensionless
(Acceleration due to gravity)$^{1/2}$, $g^{1/2}$	$(m/s^2)^{1/2}$	$(LT^{-2})^{1/2} = L^{1/2}T^{-1}$
Length of the notch, L	m	L
(Head over the sill of notch)$^{3/2}$, $H^{3/2}$	$(m)^{3/2}$	$L^{3/2}$

The dimensions of the left hand side of the equation are: L^3T^{-1}. the dimensions of the right hand side of the equation are: $(L^{1/2}T^{-1}).L.L^{3/2} = L^{1/2+1+3/2}. T^{-1} = L^3T^{-1}$

Thus we find that the dimensions of both the LHS and RHS of the equation are the same. Hence, the equation is dimensionally homogeneous.

The unique characteristic of a dimensionally homogeneous equation is that it is independent of the system of units chosen for measurement, i.e., if an equation is dimensionally homogeneous, it can be used without any modification with either system of units.

METHODS OF DIMENSIONAL ANALYSIS

(A) Rayleigh Method
(B) Buckingham π - Method

(A) Rayleigh Method

This method was proposed by Lord Rayleigh in the year 1989 to determine the effect of temperature on viscosity of a gas. Let X be a variable which is a function of different variables namely, X_1, X_2, ……, X_n. This can be written in the general form as

$$X = f\left(X_1, X_2, \ldots\ldots, X_n\right) \qquad \ldots\ldots \qquad (1)$$

In the above equation, X is the dependent variable and X_1, X_2, ……, X_n are the independent variables.

In the Rayleigh method, the functional relationship of the variables X_1, X_2, ……, X_n is expressed in the form of an exponential equation which must be dimensionally homogeneous. Hence, equation (1) can be expressed as

$$X = C\left(X_1^a X_2^b \ldots\ldots X_n^n\right) \qquad \ldots\ldots \qquad (2)$$

where C is a dimensionless constant; C can be determined either from the physical characteristics of the problem or from experimental measurements. a, b, ……, n are the exponents of X_1, X_2, ……, X_n respectively which can be evaluated on the basis that the equation is dimensionally homogeneous. By grouping together the variables with like powers, the dimensionless parameters are formed. The Rayleigh method is illustrated in the following example.

Illustration

Let us consider the problem of flow of liquid through a circular orifice discharging freely into the atmosphere under a constant head. Let Q be the discharge passing through the orifice of diameter d, under a constant head H. Let ρ be the mass density and let μ the dynamic viscosity of the liquid discharged through the orifice. Now, the discharge Q through the orifice can be assumed to be dependent on the variables namely, diameter d of the orifice, constant head H, mass density ρ of liquid, dynamic viscosity μ of liquid and the acceleration due to gravity g since the flow is freely into the atmosphere. Hence, the general functional relationship for the dependent variable Q can be written as

$$\boldsymbol{Q = f(\mu, \rho, d, H, g)} \quad \ldots\ldots \quad (3)$$

Equation (3) can be expressed by Rayleigh method in the exponential form as

$$\boldsymbol{Q = C\left(\mu^a \rho^b d^c H^d g^e\right)} \quad \ldots\ldots \quad (4)$$

where C is a dimensionless constant

The following Table shows the SI units and the dimensions of the various quantities considered in this illustration.

Quantity with symbol	**SI units**	**Dimensions (in MLT system)**
Discharge, Q	m^3s^{-1}	$M^0L^3T^{-1}$
Dynamic viscosity, μ	$kg(mass)m^{-1}s^{-1}$	$ML^{-1}T^{-1}$
Mass density, ρ	$kg(mass)m^{-3}$	$ML^{-3}T^0$
Diameter, d	m	M^0LT^0
Head, H	m	M^0LT^0
Gravitational constant, g	ms^{-2}	M^0LT^{-2}
Dimensionless constant, C	-	$M^0L^0T^0$

Substituting the dimensions for each variable in equation (4)

$$M^0L^3T^{-1} = (M^0L^0T^0)\,(ML^{-1}T^{-1})^a\,(ML^{-3}T^0)^b\,(M^0LT^0)^c\,(M^0LT^0)^d\,(M^0LT^{-2})^e$$

For dimensional homogeneity of the above equation, the exponents of each of the dimensions M, L and T on both sides of the equation must be identical. Thus

for M: $0 = a + b$ (5a)
for L: $3 = -a - 3b + c + d + e$ (5b)
for T: $-1 = -a - 2e$ (5c)

Now, there are 5 unknowns namely *a*, *b*, *c*, *d* and *e*; but there are only 3 equations; hence, three of the unknowns must be expressed in terms of the other two.

From equation (5a), $b = -a$ …… (6a)

From equation (5c), $e = \frac{1}{2} - \frac{a}{2}$ …… (6b)

From equation (5b), $3 = -a - 3(-a) + c + d + \frac{1}{2} - \frac{a}{2}$

$\rightarrow 3 = \frac{3a}{2} + c + d + \frac{1}{2}$

$\rightarrow c = \frac{5}{2} - \frac{3a}{2} - d$ …… (6c)

Substituting the values of b, c and e from equations (6a), (6c) and (6b) in (4), we have,

$$Q = C\left[\mu^a \rho^{-a} d^{\left(\frac{5}{2}-\frac{3a}{2}-d\right)} H^d g^{\left(\frac{1}{2}-\frac{a}{2}\right)}\right]$$

$$= C\left[\left(d^{\frac{5}{2}} g^{\frac{1}{2}}\right)\left(\mu^a \rho^{-a} d^{\frac{-3a}{2}} g^{\frac{-a}{2}}\right)\left(H^d d^{-d}\right)\right]$$

$$= C\left[\left(d^2 d^{\frac{1}{2}} g^{\frac{1}{2}}\right)\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^a \left(\frac{H}{d}\right)^d\right]$$

$$= C\left[\left(d^2 \frac{1}{d^{-1/2}} g^{\frac{1}{2}}\right)\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^a \left(\frac{H}{d}\right)^d\right]$$

$$= C\left[\left(d^2 g^{\frac{1}{2}}\right)\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^a \left(\frac{H}{d}\right)^d \left(\frac{1}{d}\right)^{-1/2}\right]$$

$$= C\left[\left(d^2 g^{\frac{1}{2}}\right)\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^a \left(\frac{H}{d}\right)^{d-1/2} H^{1/2}\right]$$

$$= C\left[\left(d^2 H^{1/2} g^{\frac{1}{2}}\right)\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^a \left(\frac{H}{d}\right)^{d-1/2}\right]$$

$$= \frac{C}{\frac{\pi}{4}\sqrt{2}}\left[\left(\frac{\pi}{4} d^2 \sqrt{2gH}\right)\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^a \left(\frac{H}{d}\right)^{d-1/2}\right]$$

$$= \left(a\sqrt{2gH}\right) f_1\left[\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right), \left(\frac{H}{d}\right)\right]$$

This expression may be written in the usual form as

$$Q = C_d a \sqrt{2gH} \qquad \ldots\ldots \qquad (7)$$

where C_d is the coefficient of discharge of the orifice

$$C_d = f_1\left[\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right), \left(\frac{H}{d}\right)\right] \qquad \ldots\ldots \qquad (8)$$

In the above expression, both the terms $\left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right), \left(\frac{H}{d}\right)$ are dimensionless and C_d is also a dimensionless factor.

Example 1: Show by Rayleigh method that the resistance R to the motion of a sphere of diameter D moving with a uniform velocity V through a fluid having density ρ and viscosity μ may be expressed as

$$R = \left(\rho D^2 V^2\right)\phi\left(\frac{\mu}{\rho V D}\right)$$

Solution.

The functional relationship for R may be expressed as

$$R = f(\mu, \rho, D, V) \qquad \ldots\ldots \qquad (9)$$

By Rayleigh method, equation (9) can be expressed in the exponential form as

$$R = k(\mu^a \rho^b D^c V^d) \qquad \ldots\ldots \qquad (10)$$

where k is a dimensionless constant.

The following Table shows the SI units and the dimensions of the various quantities considered in this illustration.

Quantity with symbol	SI units	Dimensions (in MLT system)
Resistance, R	*N (or)* *kg(mass)ms*$^{-2}$	$ML^{1}T^{-2}$
Dynamic viscosity, μ	*kg*(*mass*)$m^{-1}s^{-1}$	$ML^{-1}T^{-1}$
Mass density, ρ	*kg*(*mass*)m^{-3}	$ML^{-3}T^{0}$
Diameter, *D*	*m*	$M^{0}LT^{0}$
Velocity, V	ms^{-1}	$M^{0}LT^{-1}$
Dimensionless constant, *k*	-	$M^{0}L^{0}T^{0}$

Substituting the dimensions for each variable in equation (10)

$$MLT^{-2} = (M^0L^0T^0)\,(ML^{-1}T^{-1})^a\,(ML^{-3}T^0)^b\,(M^0LT^0)^c\,(M^0LT^{-1})^d$$

For dimensional homogeneity of the above equation, the exponents of each of the dimensions *M*, *L* and *T* on both sides of the equation must be identical. Thus

for *M*: $1 = a + b$ (11a)
for *L*: $1 = -a - 3b + c + d$ (11b)
for *T*: $-2 = -a - d$ (11c)

Now, there are 4 unknowns namely *a*, *b*, *c* and *d*; but there are only 3 equations; hence, three of the unknowns must be expressed in terms of the other one.

From equation (11a), $b = 1 - a$ …… (12a)
From equation (11c), $d = 2 - a$ …… (12b)
From equation (11b), $c = 1 + a + 3(1 - a) - (2 - a)$
$= 1 + a + 3 - 3a - 2 + a$
$c = 2 - a$ …… (12c)

Substituting the values of b, c and d from equations (12a), (12c) and (12b) in (10), we have,

$$R = k\left[\mu^a \rho^{(1-a)} D^{(2-a)} V^{(2-a)}\right]$$

$$= k\left[\mu^a \frac{\rho}{\rho^a}\frac{D^2}{D^a}\frac{V^2}{V^a}\right]$$

$$= k\left[\left(\rho D^2 V^2\right)\left(\frac{\mu}{\rho D V}\right)^a\right]$$

This may be expressed in the functional form as

$$R = \left(\rho D^2 V^2\right)\phi\left(\frac{\mu}{\rho D V}\right) \quad \ldots\ldots \quad (13)$$

Buckingham π - Method

Statement of Buckingham's π - Theorem: If a phenomenon is described by *n* dimensional variables, and if these *n* dimensional variables can be completely described by *m* fundamental quantities or dimensions (such as mass, length, time, etc.), and are related by a dimensionally homogeneous equation, then the relationship among the *n* quantities (or variables) can always be expressed by (*n* – *m*) dimensionless and independent π terms.
Let Y be a variable which depends on the independent variables X_1, X_2, X_3, ……, X_n. Then, the functional equation can be written as

$$Y = f(X_1, X_2, X_3, \ldots\ldots, X_n) \quad \ldots\ldots \quad (14)$$

Equation (14) can be transformed to another functional relationship as

$$f_1(Y, X_1, X_2, X_3, \ldots\ldots, X_n) = C \quad \ldots\ldots \quad (15)$$

where C is a dimensionless constant. This is as if $Y = f(X) = X^2 + C$; hence, $Y - X^2 = f_1(X, Y) = C$. In accordance with the Buckingham's π - theorem, a non-dimensional equation can be obtained as

$$f_2(\pi_1, \pi_2, \pi_3, \ldots\ldots, \pi_{n-m}) = C_1 \quad \ldots\ldots \quad (16)$$

How are these π - terms formed?

Each dimensionless π - term is formed by combining m variables out of the total n variables with one of the remaining (n – m) variables. These m variables in each of the π - terms are the same. As these m variables appear repeatedly in each of the π - terms, these variables are called repeating variables.

How are these repeating variables chosen?

These repeating variables are chosen from among the n variables such that they involve all the m fundamental quantities or dimensions and they themselves do not form any dimensionless number. Thus the different π - terms may be established as below.

$$\pi_1 = X_1^{a_1} X_2^{b_1} X_3^{c_1} \ldots\ldots X_m^{m_1} X_{m+1}$$

$$\left.\begin{array}{l}\pi_2 = X_1^{a_2} X_2^{b_2} X_3^{c_2} \ldots\ldots X_m^{m_2} X_{m+2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \pi_{n-m} = X_1^{a_{n-m}} X_2^{b_{n-m}} X_3^{c_{n-m}} \ldots\ldots X_m^{m_{n-m}} X_n\end{array}\right\} \quad \ldots\ldots \quad (17)$$

In equation (17), each individual equation is dimensionless and the exponents a, b, c, d, ……, m, etc., are determined by considering the dimensional homogeneity for each equation so that each π - term is dimensionless.

The final general equation for the phenomenon may be obtained by expressing one π - term as a function of other π - terms. That is,

$$\left.\begin{array}{l}\pi_1 = f_1(\pi_2, \pi_3, \pi_4, \ldots\ldots \pi_{n-m}) \\ \pi_2 = f_2(\pi_1, \pi_3, \pi_4, \ldots\ldots \pi_{n-m}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \pi_{n-m} = f_1(\pi_1, \pi_2, \pi_3, \ldots\ldots \pi_{n-m-1})\end{array}\right\} \quad \ldots\ldots \quad (18)$$

Illustration of Buckingham's π - method

Let us consider the same problem of flow through a small orifice as considered under the Rayleigh's method.

Step 1. The discharge of an orifice depends upon the diameter d of orifice, constant supply head H, acceleration due to gravity g, dynamic viscosity μ of liquid and mass density ρ of liquid. The functional equation for discharge Q can be written as

$$Q = f(d, H, g, \mu, \rho) \quad \ldots\ldots \quad (19)$$

Equation (19) can be expressed in its most general form as

$$f_1(Q, d, H, g, \mu, \rho) = C \quad \ldots\ldots \quad (20)$$

The total number of variables (including both the dependent variable Q and all the independent variables) n = 6. All these variables can be expressed by the three fundamental dimensions of either the M-L-T or F-L-T system. Hence, the number of fundamental quantities m= 3. Therefore, the number of dimensionless π - terms to be formed are (n –m) = (6 – 3) = 3, so that

$$f_2(\pi_1, \pi_2, \pi_3) = C_1 \quad \ldots\ldots \quad (21)$$

Step 2. Selection of Repeating Variables.

In order to form these π - terms, we have to choose m = 3 repeating variables. The criteria for choosing these m repeating variables is that these variables among themselves contain all the three fundamental dimensions and they themselves do not form any dimensionless parameter. Thus let us choose the dynamic viscosity μ with dimensions $ML^{-1}T^{-1}$, constant supply head H with dimension L and acceleration due to gravity g with dimensions LT^{-2} as the repeating variables.

Step 3. Formulation of the different π - terms.

$$\left.\begin{aligned} \pi_1 &= \mu^{a_1} H^{b_1} g^{c_1} Q \\ \pi_2 &= \mu^{a_2} H^{b_2} g^{c_2} \rho \\ \pi_3 &= \mu^{a_3} H^{b_3} g^{c_3} d \end{aligned}\right\} \quad \ldots\ldots \quad (22)$$

Step 4. Determination of the π - terms

Let us express the π_1 – term in the dimensional form using the M-L-T system.

$$\pi_1 = M^0 L^0 T^0 = \left(ML^{-1}T^{-1}\right)^{a_1} (L)^{b_1} \left(LT^{-2}\right)^{c_1} \left(L^3 T^{-1}\right)$$

Equating the exponents of M, L and T, we obtain

for M:	$0 = a_1$	……(23a)
for L:	$0 = -a_1 + b_1 + c_1 + 3$	……(23b)
for T:	$0 = -a_1 - 2c_1 - 1$	……(23c)

From (23a), $a_1 = 0$; from (23c), $c_1 = -½$; from (23b), $b_1 = -5/2$

Hence, $\pi_1 = \mu^0 H^{-5/2} g^{-1/2} Q = \dfrac{Q}{H^{5/2} g^{1/2}}$

Now, Let us express the π_1 – term in the dimensional form using the M-L-T system.

$$\pi_2 = M^0 L^0 T^0 = \left(ML^{-1}T^{-1}\right)^{a_2} (L)^{b_2} \left(LT^{-2}\right)^{c_2} \left(ML^{-3}\right)$$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_2 + 1$ ……(24a)
for L: $0 = - a_2 + b_2 + c_2 - 3$ ……(24b)
for T: $0 = - a_2 - 2c_2$ ……(24c)

From (24a), $a_2 = - 1$; from (24c), $c_2 = ½$; from (24b), $b_2 = 3/2$

Hence, $\pi_2 = \mu^{-1} H^{3/2} g^{1/2} \rho = \dfrac{\rho g^{1/2} H^{3/2}}{\mu}$

Now, Let us express the π_3 – term in the dimensional form using the M-L-T system.

$$\pi_3 = M^0 L^0 T^0 = \left(ML^{-1}T^{-1}\right)^{a_3} (L)^{b_3} \left(LT^{-2}\right)^{c_3} (L)$$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_3$ ……(25a)
for L: $0 = - a_3 + b_3 + c_3 + 1$ ……(25b)
for T: $0 = - a_3 - 2c_3$ ……(25c)

From (25a), $a_3 = 0$; from (25c), $c_3 = 0$; from (25b), $b_3 = - 1$

Hence, $\pi_3 = \mu^0 H^{-1} g^0 d = \dfrac{d}{H}$

Step 5.

As per equation (21), we have,

$$f_2(\pi_1, \pi_2, \pi_3) = C_1$$

$$\Rightarrow f_2\left(\frac{Q}{H^{5/2} g^{1/2}}, \frac{\rho g^{1/2} H^{3/2}}{\mu}, \frac{d}{H}\right) = C_1$$

or

$$\frac{Q}{H^{5/2} g^{1/2}} = C_2 f_3\left(\frac{\rho g^{1/2} H^{3/2}}{\mu}, \frac{d}{H}\right)$$

Example 1. Find the form of the equation for discharge Q through a sharp – edged triangular notch assuming Q depends on the central angle α of the notch, head H, gravitational acceleration g, and density ρ of fluid, viscosity μ of fluid and surface tension σ of fluid.

Solution.

Functional relationship for discharge Q

$$Q = f(\rho, H, g, \mu, \sigma, \alpha) \quad \ldots\ldots \quad (26)$$

Equation (26) can be written in the most general form as

$$f_1(Q, \rho, H, g, \mu, \sigma, \alpha) = C \quad \ldots\ldots \quad (27)$$

Number of variables n = 7
Number of fundamental quantities m = 3
Number of π - terms, (n – m) = 7 – 3 = 4

Repeating variables: ρ, g, H

π - terms:

$\pi_1 = \rho^{a1}. g^{b1}. H^{c1}. Q$
$\pi_2 = \rho^{a2}. g^{b2}. H^{c2}. \mu$
$\pi_3 = \rho^{a3}. g^{b3}. H^{c3}. \sigma$
$\pi_4 = \alpha$ (since, the central angle α of the notch itself is a dimensionless quantity)

π_1 – term:

$\pi_1 = \rho^{a1}. g^{b1}. H^{c1}. Q$

$$M^0L^0T^0 = (ML^{-3})^{a1} (LT^{-2})^{b1} (L)^{c1} (L^3T^{-1})$$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_1$ ……(28a)
for L: $0 = -3a_1 + b_1 + c_1 + 3$ ……(28b)
for T: $0 = -2b_1 - 1$ ……(28c)

From (28a), $a_1 = 0$; from (28c), $b_1 = -½$; from (28b), $c_1 = -5/2$

$$\pi_1 = \frac{Q}{g^{1/2} H^{5/2}}$$

π_2 – term:

$\pi_2 = \rho^{a2}.\ g^{b2}.\ H^{c2}.\ \mu$

$M^0L^0T^0 = (ML^{-3})^{a2}\ (LT^{-2})^{b2}\ (L)^{c2}\ (ML^{-1}T^{-1})$

Equating the exponents of M, L and T, we obtain

for M:	$0 = a_2 + 1$	……(29a)
for L:	$0 = -3a_2 + b_2 + c_2 - 1$	……(29b)
for T:	$0 = -2b_2 - 1$	……(29c)

From (29a), $a_2 = -1$; from (29c), $b_2 = -½$; from (29b), $c_2 = -3/2$

$$\pi_2 = \frac{\mu}{\rho g^{1/2} H^{3/2}}$$

π_3 – term:

$\pi_3 = \rho^{a3}.\ g^{b3}.\ H^{c3}.\ \sigma$

$M^0L^0T^0 = (ML^{-3})^{a3}\ (LT^{-2})^{b3}\ (L)^{c3}\ (MT^{-2})$

Equating the exponents of M, L and T, we obtain

for M:	$0 = a_3 + 1$	……(30a)
for L:	$0 = -3a_3 + b_3 + c_3$	……(30b)
for T:	$0 = -2b_3 - 2$	……(30c)

From (30a), $a_3 = -1$; from (30c), $b_3 = -1$; from (30b), $c_3 = -2$

$$\pi_3 = \frac{\sigma}{\rho g H^2}$$

Thus we have,

$$f_2(\pi_1, \pi_2, \pi_3, \pi_4) = C_1$$

$$\Rightarrow f_2\left(\frac{Q}{g^{1/2} H^{5/2}}, \frac{\mu}{\rho g^{1/2} H^{3/2}}, \frac{\sigma}{\rho g H^2},\right) = C_1$$

$$\Rightarrow \frac{Q}{g^{1/2}H^{5/2}} = C_2 f_3\left(\frac{\mu}{\rho g^{1/2}H^{3/2}}, \frac{\sigma}{\rho g H^2}, \alpha\right)$$

$$\Rightarrow Q = g^{1/2}\phi\left(\frac{\mu}{\rho g^{1/2}H^{3/2}}, \frac{\sigma}{\rho g H^2}, \alpha\right)H^{5/2}$$

$$\Rightarrow Q = CH^{5/2}$$

where $C = g^{1/2}\phi\left(\frac{\mu}{\rho g^{1/2}H^{3/2}}, \frac{\sigma}{\rho g H^2}, \alpha\right)$

Example 2. The discharge Q of a centrifugal pump is dependent on the mass density ρ of fluid, pump speed N (rpm), diameter D of impeller, pressure p of fluid, and viscosity μ of fluid. Show using Buckingham's π - theorem that Q can be represented by

$$Q = (ND^3)\phi\left(\frac{gH}{N^2D^2}, \frac{\upsilon}{ND^2}\right)$$

where H = head and υ = kinematic viscosity of the fluid.

Solution.

Functional relationship for discharge Q

$$Q = f(\rho, N, D, p, \mu) \qquad \ldots\ldots \qquad (31)$$

Equation (31) can be written in the most general form as

$$f_1(Q, \rho, N, D, p, \mu) = C \qquad \ldots\ldots \qquad (32)$$

Number of variables n = 6
Number of fundamental quantities m in which the six variables can be expressed = 3
Number of π - terms, $(n - m) = 7 - 3 = 4$

Repeating variables: ρ, N, D

π - terms:

$\pi_1 = \rho^{a1}. N^{b1}. D^{c1}. Q$
$\pi_2 = \rho^{a2}. N^{b2}. D^{c2}. p$
$\pi_3 = \rho^{a3}. N^{b3}. D^{c3}. \mu$

π_1 – term:

$\pi_1 = \rho^{a1}. N^{b1}. D^{c1}. Q$

$M^0L^0T^0 = (ML^{-3})^{a1} (T^{-1})^{b1} (L)^{c1} (L^3T^{-1})$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_1$ ……(33a)
for L: $0 = - 3a_1 + c_1 + 3$ ……(33b)
for T: $0 = - b_1 – 1$ ……(33c)

From (33a), $a_1 = 0$; from (33c), $b_1 = - 1$; from (33b), $c_1 = - 3$

$$\pi_1 = \frac{Q}{ND^3}$$

π_2 – term:

$\pi_2 = \rho^{a2}. N^{b2}. D^{c2}. p$

$M^0L^0T^0 = (ML^{-3})^{a2} (T^{-1})^{b2} (L)^{c2} (ML^{-1}T^{-2})$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_2 + 1$ ……(34a)
for L: $0 = - 3a_2 + c_2 – 1$ ……(34b)
for T: $0 = - b_2 – 2$ ……(34c)

From (34a), $a_2 = - 1$; from (34c), $b_2 = - 2$; from (34b), $c_2 = - 2$

$$\pi_2 = \frac{p}{\rho N^2 D^2}$$

π_3 – term:

$\pi_3 = \rho^{a3}. N^{b3}. D^{c3}. \mu$

$M^0L^0T^0 = (ML^{-3})^{a3} (T^{-1})^{b3} (L)^{c3} (ML^{-1}T^{-1})$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_3 + 1$ ……(35a)
for L: $0 = -3a_3 + c_3 - 1$ ……(35b)
for T: $0 = -b_3 - 1$ ……(35c)

From (35a), $a_3 = -1$; from (35c), $b_3 = -1$; from (35b), $c_3 = -2$

$$\pi_3 = \frac{\mu}{\rho N D^2}$$

Thus we have,

$$f_2(\pi_1, \pi_2, \pi_3) = C_1$$
$$f_2\left(\frac{Q}{ND^3}, \frac{p}{\rho N^2 D^2}, \frac{\mu}{\rho N D^2}\right) = C_1$$
$$\Rightarrow \frac{Q}{ND^3} = C_2 f_3\left(\frac{p}{\rho N^2 D^2}, \frac{\mu}{\rho N D^2}\right)$$
$$\Rightarrow \frac{Q}{ND^3} = \phi\left(\frac{p}{\rho N^2 D^2}, \frac{\mu}{\rho N D^2}\right)$$

Since, $p = \rho gH$ and $\frac{\mu}{\rho} = \upsilon$, we have,

$$\Rightarrow \frac{Q}{ND^3} = \phi\left(\frac{gH}{N^2 D^2}, \frac{\upsilon}{ND^2}\right)$$
$$\Rightarrow Q = \left(ND^3\right)\phi\left(\frac{gH}{N^2 D^2}, \frac{\upsilon}{ND^2}\right)$$

Example 3. Show by π-theorem that a general equation for discharge Q over a weir of any shape is given by

$$Q = (H^{5/2} g^{1/2})\phi\left[\left(\frac{\upsilon}{H^{3/2} g^{1/2}}\right), \left(\frac{\sigma}{H^2 \rho g}\right)\right]$$

where H = head over the weir, ν = kinematic viscosity of the liquid, ρ = mass density of the liquid, and σ = surface tension of the liquid. Hence show that discharge over a rectangular weir of crest length L is given by

$$Q = C_d L H^{3/2}$$

Solution:

Functional relationship for discharge Q

$Q = f\,(H, g, \nu, \rho, \sigma)$ …… (36)

Equation (36) can be written in the most general form as

$f_1 (Q, H, g, \nu, \rho, \sigma) = C$ …… (37)

Number of variables $n = 6$

Number of fundamental quantities m in which the six variables can be expressed = 3

Number of π - terms, $(n – m) = 6 – 3 = 3$

Repeating variables: ρ, g, H

π - terms:

$\pi_1 = \rho^{a1}. g^{b1}. H^{c1}. Q$
$\pi_2 = \rho^{a2}. g^{b2}. H^{c2}. \nu$
$\pi_3 = \rho^{a3}. g^{b3}. H^{c3}. \sigma$

π_1 – term:

$\pi_1 = \rho^{a1}. g^{b1}. H^{c1}. Q$
$M^0L^0T^0 = (ML^{-3})^{a1} (LT^{-2})^{b1} (L)^{c1} (L^3T^{-1})$

Equating the exponents of M, L and T, we obtain
for M: $0 = a_1$ ……(38*a*)
for L: $0 = - 3a_1 + b_1 + c_1 + 3$ ……(38*b*)
for T: $0 = - 2b_1 – 1$ ……(38*c*)

From (38*c*), $b_1 = -1/2$
Putting $a_1 = 0$ and $b_1 = -1/2$ in (38*b*), we have,
$c_1 = -5/2$

Hence, $\pi_1 = \rho^0. g^{-1/2}. H^{-5/2}. Q$

$$= \frac{Q}{g^{1/2} H^{5/2}}$$

π_2 – term:

$\pi_2 = \rho^{a2}. g^{b2}. H^{c2}. \nu$
$M^0L^0T^0 = (ML^{-3})^{a2} (LT^{-2})^{b2} (L)^{c2} (L^2T^{-1})$

Equating the exponents of M, L and T, we obtain
for M: $0 = a_2$ ……(39*a*)
for L: $0 = - 3a_2 + b_2 + c_2 + 2$ ……(39*b*)

for T: $0 = -2b_2 - 1$ ……(39*c*)

From (39*c*), $b_2 = -1/2$
Putting $a_2 = 0$ and $b_2 = -1/2$ in (39*c*), $c_2 = -3/2$

Hence, $\pi_2 = \rho^0 . g^{-1/2} . H^{-3/2} . \nu$

$$= \frac{\upsilon}{g^{1/2} H^{3/2}}$$

π_3 – term:

$\pi_3 = \rho^{a3} . g^{b3} . H^{c3} . \sigma$
$M^0 L^0 T^0 = (ML^{-3})^{a3} (LT^{-2})^{b3} (L)^{c3} (MT^{-2})$

Equating the exponents of M, L and T, we obtain
for M: $0 = a_3 + 1$ ……(40*a*)
for L: $0 = -3a_3 + b_3 + c_3$ ……(40*b*)
for T: $0 = -2b_3 - 2$ ……(40*c*)

From (40*a*), $a_3 = -1$; from (40*c*), $b_3 = -1$
Putting $a_3 = -1$ and $b_3 = -1$ in (40*b*), we have, $c_3 = -2$

Hence, $\pi_3 = \rho^{-1} . g^{-1} . H^{-2} . \sigma$

$$= \frac{\sigma}{\rho g H^2}$$

Thus we have,

$$f_2(\pi_1, \pi_2, \pi_3) = C_1$$

$$\Rightarrow f_2\left[\left(\frac{Q}{g^{1/2} H^{5/2}}\right), \left(\frac{\upsilon}{g^{1/2} H^{3/2}}\right), \left(\frac{\sigma}{\rho g H^2}\right)\right] = C_1$$

$$\Rightarrow \left(\frac{Q}{g^{1/2} H^{5/2}}\right) = C_2 f_3 \left[\left(\frac{\upsilon}{g^{1/2} H^{3/2}}\right), \left(\frac{\sigma}{\rho g H^2}\right)\right]$$

$$\Rightarrow Q = (g^{1/2} H^{5/2}) \phi \left[\left(\frac{\upsilon}{g^{1/2} H^{3/2}}\right), \left(\frac{\sigma}{\rho g H^2}\right)\right]$$

We have, $\frac{Q}{g^{1/2} H^{5/2}}$ = a dimensionless term; this can be written as

$\frac{Q}{g^{1/2}HH^{3/2}} = \frac{Q}{g^{1/2}LH^{3/2}}$ which is also dimensionless (the term *H* is replaced by the crest length *L* of the rectangular weir)

Let $\frac{Q}{g^{1/2}LH^{3/2}} = C$

$\Rightarrow \frac{Q}{LH^{3/2}} = Cg^{1/2} = C_d$

$\Rightarrow Q = C_d LH^{3/2}$

Example 4. By dimensional analysis show that the torque *T* on a shaft of diameter *d*, revolving at a speed *N* in a fluid of viscosity μ and mass density ρ is given by the expression

$$T = \left(\rho d^5 N^2\right)\phi\left(\frac{\upsilon}{d^2 N}\right)$$

Use Buckingham's method. Hence show that power *P* is given by

$$P = \left(\rho d^5 N^3\right)\phi\left(\frac{\upsilon}{d^2 N}\right)$$

Solution:

Functional relationship for discharge *T*

$T = f\,(d, N, \mu, \rho,)$ …… (41)

Equation (36) can be written in the most general form as

$f_1\,(T, d, N, \mu, \rho,) = C$ …… (42)

Number of variables $n = 5$

Number of fundamental quantities *m* in which the six variables can be expressed = 3

Number of π - terms, $(n - m) = 5 - 3 = 2$

Repeating variables: ρ, *d*, *N*

π - terms:

$\pi_1 = \rho^{a1}.\ d^{b1}.\ N^{c1}.\ T$

$\pi_2 = \rho^{a2}.\ d^{b2}.\ N^{c2}.\ \mu$

π_1 - term:

$\pi_1 = \rho^{a1}.\ d^{b1}.\ N^{c1}.\ T$
$M^0L^0T^0 = (ML^{-3})^{a1}\ (L)^{b1}\ (T^{-1})^{c1}\ (ML^2T^{-2})$

Equating the exponents of M, L and T, we obtain
for M: $0 = a_1 + 1$ ……(43a)
for L: $0 = -3a_1 + b_1 + 2$ ……(43b)
for T: $0 = -c_1 - 2$ ……(43c)

From (43a), $a_1 = -1$; from (43c), $c_1 = -2$; putting $a_1 = -1$ in (43b), we have, $b_1 = -5$

Hence, $\pi_1 = \rho^{-1}.\ d^{-5}.\ N^{-2}.\ T$

$$\pi_1 = \left(\frac{T}{\rho d^5 N^2}\right)$$

π_2 - term:

$\pi_2 = \rho^{a2}.\ d^{b2}.\ N^{c2}.\ \mu$
$M^0L^0T^0 = (ML^{-3})^{a2}\ (L)^{b2}\ (T^{-1})^{c2}\ (ML^{-1}T^{-1})$

Equating the exponents of M, L and T, we obtain
for M: $0 = a_2 + 1$ ……(43a)
for L: $0 = -3a_2 + b_2 - 1$ ……(43b)
for T: $0 = -c_2 - 1$ ……(43c)

From (43a), $a_2 = -1$; from (43c), $c_2 = -1$; putting $a_2 = -1$ in (43b), we have, $b_2 = -2$

Hence, $\pi_2 = \rho^{-1}.\ d^{-2}.\ N^{-1}.\ \mu$

$$\pi_2 = \left(\frac{\mu}{\rho d^2 N}\right)$$

As $\frac{\mu}{\rho} = \upsilon$, where υ is the kinematic viscosity, the π_2 – term can be written as

$$\pi_2 = \left(\frac{\upsilon}{d^2 N}\right)$$

Thus we have,

$$f_2(\pi_1, \pi_2) = C_1$$

$$f_2\left[\left(\frac{T}{\rho d^5 N^2}\right),\left(\frac{\upsilon}{d^2 N}\right)\right]=C_1$$

$$\Rightarrow\left(\frac{T}{\rho d^5 N^2}\right)=C_2 f_3\left(\frac{\upsilon}{d^2 N}\right)$$

$$\Rightarrow T=(\rho d^5 N^2)\phi\left(\frac{\upsilon}{d^2 N}\right)$$

We know that power, P = (Torque, T) x (angular velocity, ω)

Angular velocity, $\omega = \frac{2\pi N}{60}$

Hence, $P = \frac{2\pi N}{60}(\rho d^5 N^2)\phi\left(\frac{\upsilon}{d^2 N}\right) = (\rho d^5 N^3)\phi\left(\frac{\upsilon}{d^2 N}\right)$

It should be noted that in the above expression, the quantity $\left(\frac{P}{\rho d^5 N^3}\right)$ is a dimensionless quantity

Example 5. The resistance R to the motion of a supersonic aircraft of length L, moving with a velocity V in air of density ρ, depends on the viscosity μ and bulk modulus of elasticity K of air. Obtain using Buckingham's π - theorem, the following expression for the resistance R

$$R=\left(\rho L^2 V^2\right)\phi\left[\left(\frac{\mu}{\rho L V}\right),\left(\frac{K}{\rho V^2}\right)\right]$$

Solution:

Functional relationship for discharge R

$R = f(L, V, \mu, \rho, \text{K})$ …… (44)

Equation (44) can be written in the most general form as

$f_1(R, L, V, \mu, \rho, \text{K}) = C$ …… (45)

Number of variables $n = 6$

Number of fundamental quantities m in which the six variables can be expressed = 3

Number of π - terms, $(n - m) = 6 - 3 = 3$

Repeating variables: ρ, L, V

π - terms:

$\pi_1 = \rho^{a1}. L^{b1}. V^{c1}. R$
$\pi_2 = \rho^{a2}. L^{b2}. V^{c2}. \mu$
$\pi_3 = \rho^{a3}. L^{b3}. V^{c3}. K$

π_1 - term:

$\pi_1 = \rho^{a1}. L^{b1}. V^{c1}. R$
$M^0L^0T^0 = (ML^{-3})^{a1} (L)^{b1} (LT^{-1})^{c1} (MLT^{-2})$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_1 + 1$ ……(46*a*)
for L: $0 = - 3a_1 + b_1 + c_1 + 1$ ……(46*b*)
for T: $0 = - c_1 – 2$ ……(46*c*)

From (46*a*), $a_1 = - 1$; from (46*c*), $c_1 = - 2$; putting $a_1 = - 1$ and $c_1 = - 2$ in (46*b*), we have, $b_1 = - 2$

Hence, $\pi_1 = \rho^{-1}. L^{-2}. V^{-2}. R$

$$\pi_1 = \left(\frac{R}{\rho L^2 V^2}\right)$$

π_2 - term:

$\pi_2 = \rho^{a2}. L^{b2}. V^{c2}. \mu$
$M^0L^0T^0 = (ML^{-3})^{a2} (L)^{b2} (LT^{-1})^{c2} (ML^{-1}T^{-1})$

Equating the exponents of M, L and T, we obtain

for M: $0 = a_2 + 1$ ……(47*a*)
for L: $0 = - 3a_2 + b_2 + c_2 - 1$ ……(47*b*)
for T: $0 = - c_2 – 1$ ……(47*c*)

From (47*a*), $a_2 = - 1$; from (47*c*), $c_2 = - 1$; putting $a_2 = - 1$ and $c_2 = - 1$ in (47*b*), we have, $b_2 = - 1$

Hence, $\pi_2 = \rho^{-1}. L^{-1}. V^{-1}. \mu$

$$\pi_2 = \left(\frac{\mu}{\rho L V}\right)$$

π_3 - term:

$\pi_3 = \rho^{a3}. L^{b3}. V^{c3}. K$
$M^0L^0T^0 = (ML^{-3})^{a3} (L)^{b3} (LT^{-1})^{c3} (ML^{-1}T^{-2})$

Equating the exponents of M, L and T, we obtain

or M: $0 = a_3 + 1$ ……(48a)
for L: $0 = -3a_3 + b_3 + c_3 - 1$ ……(48b)
for T: $0 = -c_3 - 2$ ……(48c)

From (48a), $a_3 = -1$; from (48c), $c_3 = -2$; putting $a_3 = -1$ and $c_3 = -2$ in (48b), we have, $b_3 = 0$

Hence, $\pi_3 = \rho^{-1}. L^0. V^{-2}. K$

$$\pi_3 = \left(\frac{K}{\rho V^2}\right)$$

Hence, $f_2(\pi_1, \pi_2, \pi_3) = C_1$

$$\Rightarrow f_2\left[\left(\frac{R}{\rho L^2 V^2}\right), \left(\frac{\mu}{\rho L V}\right), \left(\frac{K}{\rho V^2}\right)\right] = C_1$$

$$\Rightarrow \left(\frac{R}{\rho L^2 V^2}\right) = C_2\, f_3\left[\left(\frac{\mu}{\rho L V}\right), \left(\frac{K}{\rho V^2}\right)\right]$$

$$\Rightarrow R = \left(\rho L^2 V^2\right)\phi\left[\left(\frac{\mu}{\rho L V}\right), \left(\frac{K}{\rho V^2}\right)\right]$$

MODEL INVESTIGATION

Different kinds of hydraulic structures such as dams, spillways, canal head works and diversion structures and hydraulic machines such as turbines and pumps are designed and constructed to yield efficiently the desired output. We have to ascertain that the designed structures after construction will definitely yield the desired output. In case, if the structure is found not to perform as per the design, then it is not possible to rectify the same as the structures are very massive. Therefore, it becomes imperative to study, in advance, how the structure or the machine would perform once it is constructed. For this purpose, one has to resort to experimental investigation. As the real structures or machines to be constructed in the field are very huge in size, it is not feasible to conduct these experimental investigations on the full size of the structure.

Hence, a small scale replica of the actual structure is constructed and then tests are conducted to obtain the desired information. The small scale replica (imitation) of the actual structure or the machine is known as its ***model*** whiles the actual structure or machine is known as the ***prototype***.

The model tests are quite economical and convenient as the design, construction and operation of the model may be altered several times if necessary, till all the discrepancies found in the model are eliminated and the most suitable design is obtained. On the basis of final results obtained from the model tests, the design of the prototype may be modified and also it may be possible to predict the behaviour of the prototype. However, the model test results can be used to obtain the performance of the prototype only if a complete similarity between the model and the prototype exists. This may be achieved as below.

TYPES OF SIMILARITIES

There are three types of similarities to be established for complete similarity to exist between the model and the prototype. They are:

1. Geometric similarity
2. Kinematic similarity
3. Dynamic similarity

1. Geometric similarity:

When a model and the corresponding prototype are said be geometrically similar?

If the ratios of the corresponding length dimensions of the model and the prototype are equal, the model and the prototype are said to be geometrically similar. Such a ratio is called scale ratio.

For example, let the lengths, breadths and depths of a model and the corresponding prototype be respectively, L_m, b_m and d_m and L_p, b_p and d_p.

Then the length scale ratios are: L_m / L_p, b_m / b_p and d_m / d_p. If these scale ratios are equal, then the model and the prototype are said to be geometrically similar.

Hence, for geometric similarity between the model and the prototype,

Length scale ratio, $L_r = L_m / L_p = b_m / b_p = d_m / d_p$

The area scale ratio, A_r, is defined as the ratio of the area of the model and the area of the prototype.

$$A_r = A_m / A_p = (L_m \text{ x } b_m) / (L_p \text{ x } b_p) = (L_m / L_p)\ (b_m / b_p) = L_r.\ L_r = L_r^2$$

Similarly, the volume scale ratio, V_r, is defined as the ratio of the volume of the model and the volume of the prototype.

$$V_r = V_m / V_p = (L_m \text{ x } b_m \text{ x } d_m) / (L_p \text{ x } b_p \text{ x } d_p) = (L_m / L_p)\ (b_m / b_p)\ (d_m / d_p)$$
$$= L_r.\ L_r.L_r = L_r^3$$

It is thus observed that, if the model and the prototype are geometrically similar, by mere change of the scale, both the model and the prototype can be superimposed.

2. Kinematic similarity:

When a model and the corresponding prototype are said be kinematically similar?

If the paths of the homologous moving particles are geometrically similar and if the ratios of the velocities as well as accelerations of the homologous particles are equal, kinematic similarity is said to exist between the model and the prototype.
What is a homologous point?

Consider a model and the corresponding prototype which possess geometric similarity. A point in the model and the corresponding point in the prototype are said to be homologous points.

Since, both velocity and acceleration are vector quantities (i.e., both have magnitude and direction), kinematic similarity implies that the directions of velocities and accelerations at corresponding points (i.e., homologous points) are parallel to each other and the ratios of magnitudes of both velocities and accelerations at corresponding points in the model and the prototype have constant values at all corresponding set of points. Some of the scale ratios which are useful in describing kinematic similarity are:

Time scale ratio, $T_r = T_m / T_p$

Velocity scale ratio, $V_r = V_m / V_p = (L_m / T_m) / (L_p / T_p)$
$$= (L_m / L_p)\ (T_p / T_m)$$
$$= (L_m / L_p)\ \{1 / (T_m / T_p)\}$$

$= L_r / T_r$

where, V_m and V_p are respectively, the velocities of flow in the model and the prototype at homologous points.

Acceleration scale ratio, $a_r = a_m / a_p = \{L_m / (T_m)^2\} / \{L_p / (T_p)^2\}$
$= (L_m / L_p) \{(T_p)^2 / (T_m)^2\}$
$= (L_m / L_p) [1 / \{(T_m)^2 / (T_p)^2\}]$
$= L_r / T_r^2$

Discharge scale ratio, $\underline{Q_r} = Q_m / Q_p = \{(L_m)^3 / T_m\} / \{(L_p)^3 / T_p\}$
$= \{(L_m)^3 / (L_p)^3\}\{T_p / T_m\}$
$= \{(L_m)^3 / (L_p)^3\} \{1 / (T_m / T_p)\}$
$= L_r^3 / T_r$

Kinematic similarity can be attained if the flow nets for the model and the prototype are geometrically similar. This means that, by mere change of the scale, the flow net for the model and the flow net for the prototype can be superimposed.

3. Dynamic similarity:

When a model and the corresponding prototype are said be dynamically similar?

If the ratios of all the forces acting at homologous points in the model and the prototype which possess both geometric and kinematic similarities are equal, then it is said that the model and the prototype possess dynamic similarity.

In the problems concerning fluid flow, the forces acting may be any one, or a combination of several of the following forces:

(i) Inertia forces, F_i
(ii) Friction or viscous forces, F_v
(iii) Gravity forces, F_g
(iv) Pressure forces, F_p
(v) Elastic forces, F_e
(vi) Surface tension forces, F_s

Inertia force, F_i, is the force of resistance offered by an inert mass to acceleration. According to Newton's law of motion, the magnitude of the inertial force is equal to the product of the particle mass and acceleration of the particle. The direction of the inertia force is opposite to the direction of the acceleration of the particle.

The conditions required for complete dynamic similarity are developed from the Newton's Second Law of Motion. In a flowing fluid, if a fluid particle of mass M is subjected to acceleration a, then the inertial force F_i of the particle equals 'Ma'. If all the above listed forces come into play in the fluid flow system under consideration, then the resultant force, $\sum F$, which is the vectorial sum of all the listed forces acting on the fluid particle, will be equal to the inertial force of the fluid particle, i.e.,

$$\sum F = F_v + F_g + F_p + F_e + F_s = Ma$$

For complete dynamic similarity to exist between the model and its prototype, the ratio of the inertia forces of the model and the prototype must be equal tot he ratio of the resultant forces of the model and the prototype. i.e.,

$$(\sum F)_m / (\sum F)_p = (F_v + F_g + F_p + F_e + F_s)_m / (F_v + F_g + F_p + F_e + F_s)_p$$
$$= (Ma)_m / (Ma)_p \quad \ldots\ldots \quad (1)$$

In addition to the above stated condition for complete dynamic similarity, the ratio of the inertia forces of the model and the prototype must also be equal to the ratios of the individual component forces of the model and the prototype. i.e.,

(i) $(F_v)_m / (F_v)_p = (Ma)_m / (Ma)_p$
(or)
$$(Ma/F_v)_m = (Ma/F_v)_p \quad \ldots\ldots \quad (2)$$

(ii) $(F_g)_m / (F_g)_p = (Ma)_m / (Ma)_p$
(or)
$$(Ma/F_g)_m = (Ma/F_g)_p \quad \ldots\ldots \quad (3)$$

(iii) $(F_p)_m / (F_p)_p = (Ma)_m / (Ma)_p$
(or)
$$(Ma/F_p)_m = (Ma/F_p)_p \quad \ldots\ldots \quad (4)$$

(iv) $(F_e)_m / (F_e)_p = (Ma)_m / (Ma)_p$
(or)
$$(Ma/F_v)_m = (Ma/F_v)_p \quad \ldots\ldots \quad (5)$$

(v) $(F_s)_m / (F_s)_p = (Ma)_m / (Ma)_p$
(or)
$$(Ma/F_s)_m = (Ma/F_s)_p \quad \ldots\ldots \quad (6)$$

Thus, it may be mentioned that when both the model and the prototype are geometrically, kinematically and dynamically similar, then they are said to be completely similar or complete similitude exists between the two systems. However, the existence of dynamic similarity implies that both geometric and kinematic similarities exist between the model and the prototype. Hence, if dynamic similarity exists between the model and the prototype, they are said to be completely similar. Further, for complete similarity to exist between the model and the prototype, the dimensionless terms (or the π - terms) formed from the complete set of variables involved must be the same for both the model and the prototype.

DIMENSIONLESS NUMBERS (FORCE RATIOS)

When a mass is in motion, inertial force always exists. Hence, in order to develop the conditions for dynamic similarity, the ratio of inertial force and any one of the remaining forces listed previously is considered. Each of these ratios will obviously be a non-dimensional factor. The various force ratios are discussed herein:

(a) Inertia force – viscous force ratio (Reynolds number)

We know that, Inertia force = mass x acceleration

Since, mass density ρ = mass / volume, mass can be expressed as the product of mass density ρ and volume. Acceleration is the rate of change of velocity. Hence, we have,

Inertia force = (mass density x volume) (velocity / time)
= mass density x (volume / time) x velocity

By definition, (volume / time) represents the discharge. Discharge is the product of cross-sectional area of flow, A and the velocity of flow, V, i.e., discharge, $Q = AV$

So, Inertia force = $\rho (AV) V = \rho AV^2$

As cross sectional area of flow passage, A, has dimensions of L^2, we have,

Inertia force, $F_i = \rho L^2 V^2$

By definition, as per Newton's law of viscosity, we have, shear stress due to viscous force, F_v, is given by

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$\tau = \mu (dV / dy)$

where, μ = coefficient of viscosity of fluid (or) simply, the dynamic viscosity of the fluid

(dV / dy) = velocity gradient

Viscous force, F_v = shear stress x area = $\tau A = \mu (dV / dy) A$

Assuming (dV / dy) to be linear, the above expression can be written as

$F_v = \mu (V / y) A$

As y represents the thickness of fluid film, it has dimensions of L. The dimension of the area 'A' is 'L^2'. Replacing 'y' by 'L' and 'A' by 'L^2' , the above expression for F_v becomes

$F_v = \mu (V / L) L^2 = \mu VL$

Now, the ratio between the inertia force, F_i, and the viscous force, F_v, is given by

$(F_i / F_v) = (\rho L^2 V^2) / (\mu VL) = (\rho LV / \mu) = (VL / \nu)$

where ν is the kinematic viscosity of the fluid.

The force ratio (or) non-dimensional ratio, $(\rho LV / \mu)$, is called the ***Reynolds number***, *Re* or N_R.

The Reynolds number indicates the relative predominance of the inertia force to the viscous force occurring in the flow system. If the Reynolds number is larger, greater will be the relative magnitude of inertia force. If the Reynolds number is smaller, the greater will be the relative magnitude of viscous force.

(b) Inertia force – Gravity force ratio (Froude number)

From the previous discussion, we have, $F_i = \rho L^2 V^2$

As per Newton's second law of motion, force due to gravity can be expressed as

F_g = mass x acceleration due to gravity

Mass can be expressed as the product of mass density, ρ and the volume; hence,

F_g = (mass density x volume) (acceleration due to gravity)

= (ρ x volume) x g

Volume has dimensions of L^3. Replacing 'volume' by 'L^3', the above expression becomes

$F_g = \rho L^3 g$

Now, the ratio between the inertia force, F_i, and the gravity force, F_g, is given by

$(F_i / F_g) = (\rho L^2 V^2) / (\rho L^3 g) = V^2 / Lg$

The square root of this ratio, i.e., $(V^2 / Lg)^{1/2} = V / (Lg)^{1/2}$ is called the ***Froude number.***

(c) Inertia Force – Pressure Force ratio (Euler number)

Pressure force, F_p can be expressed as the product of the pressure intensity, p and the area, A, over which it acts. i.e.,

$F_p = p \times A$

Area A has dimensions of L^2; Replacing 'A' by 'L^2', the above expression becomes

$F_p = p \times L^2$

Hence, the ratio between the inertia force, F_i, and the pressure force, F_p, is given by

$F_i / F_p = (\rho L^2 V^2) / (pL^2) = \rho V^2 / p = V^2 / (p / \rho)$

The square root of this ratio, i.e., $[V^2 / (p / \rho)]^{1/2} = [V / (p/\rho)^{1/2}]$ is called the ***Euler number***, Eu or N_E. The reciprocal of Euler number, i.e., $[(p / \rho)^{1/2} / V]$ is sometimes known as 'Newton number'.

(d) Inertia force – Elasticity force ratio (Mach number)

Force due to elasticity, F_e, is expressed as the product of the bulk modulus of elasticity, K, of the flowing fluid and the area, A, over which the force acts, i.e.,

$F_e = K \times A$

As the dimensions of area, A, are L^2, the above expression becomes

$F_e = K \times L^2$

The ratio between the inertia force, F_i, and the force due to elasticity, F_e, is given by

$F_i / F_e = (\rho L^2 V^2) / (KL^2) = \rho V^2 / K = V^2 / (K / \rho) = V^2 / C^2$

where, $C = (K / \rho)^{1/2}$ = velocity of sound in that fluid medium whose bulk modulus of elasticity, K, and mass density, ρ, are being considered.
The ratio (V^2 / C^2) is known as the ***'Cauchy number'***. The square root of this ratio, i.e., (V / C) or $\{V / (K / \rho)^{1/2}\}$ is known as the ***'Mach number'***, Ma or N_M. When $Ma > 1$, i.e., $V > C$, or in other words, the characteristic velocity of flow of the fluid is more than velocity of sound in that flow medium, the flow is said to be supersonic. When $Ma < 1$, i.e., $V < C$, or in other words, the characteristic velocity of flow of the fluid is less than velocity of sound in that flow medium, the flow is said to be subsonic. When $Ma = 1$, or $V = C$, the flow is considered to be sonic. When $Ma >> 1$, i.e., $V >> C$, then the flow is sometimes termed as hypersonic. A higher Mach number indicates the predominance of the effect of compressibility of the fluid. However, when the Mach number is relatively small, say, less than 0.4, the effect of compressibility of the fluid can be neglected.

(e) Inertia force – Surface tension force ratio (Weber number)

Force due to surface tension, $F_s = \sigma L$

where σ = surface tension of fluid in contact with, say, air (in N/m)
L = length of the fluid film over which the force due to surface tension acts

Hence, the ratio of the inertia force, F_i, and the surface tension force, F_s, is given by

$F_i / F_s = (\rho L^2 V^2) / (\sigma L) = (\rho L V^2) / \sigma = [V^2 / \{\sigma / (\rho L)\}]$

The square root of this ratio, i.e., $[V / \{\sigma / (\rho L)\}^{1/2}]$ is called the ***Weber number***.

SIMILARITY LAWS OR MODEL LAWS

The results obtained from the model tests can be transferred to the prototype by the use of model laws. The model laws can be developed from the principles of dynamic similarity. The conditions for the existence of dynamic similarity between the model and the prototype are depicted by equations (1) to (6). In almost all hydraulic problems encountered in practice, for which model studies are required to be carried out, it is quite rare that all the forces, namely, F_i, F_g, F_v, F_p, F_e and F_s are simultaneously predominant in the flow phenomenon. Moreover, in most of the fluid flow problems, only one force in addition to the inertia force, F_i, is relatively more significant than the rest of the forces. The rest of the forces may either do not exist or may be of negligible magnitude. Under these circumstances, the various model laws have been developed depending upon the significant influence of each of the forces on the different fluid flow phenomena. In the derivation of these model laws, it has been assumed that for equal values of the dimensionless parameters the corresponding flow pattern in model and its prototype are similar.

(a) Reynolds Model Law

In case of flows where, in addition to the inertia force, the only other force of significance is the viscous force, the similarity in flow in the model and the prototype can be obtained if the Reynolds number of flow is the same in both the model and the prototype. This is known as ***Reynolds Model Law***.

According to the law, we have,

$$(N_R)_{model} = (N_R)_{prototype}$$

$$(\rho_m V_m L_m) / \mu_m = (\rho_p V_p L_p) / \mu_p$$

where $(N_R)_{model}$ = Reynolds number of flow in model

$(N_R)_{prototype}$ = Reynolds number of flow in prototype

ρ_m = mass density of fluid in model

V_m = characteristic velocity of flow in model

L_m = characteristic length in model

μ_m = dynamic viscosity of fluid in model

ρ_p = mass density of fluid in prototype

V_p = characteristic velocity of flow in prototype

L_p = characteristic length in prototype

μ_p = dynamic viscosity of fluid in prototype

Dividing LHS by RHS of above equation,

$$[(\rho_m V_m L_m) / \mu_m] / [(\rho_p V_p L_p) / \mu_p] = (\rho_m / \rho_p)(V_m / V_p)(L_m / L_p)(\mu_p / \mu_m)$$

$$= \{(\rho_m / \rho_p)(V_m / V_p)(L_m / L_p)\} / \{(\mu_m / \mu_p)\}$$

$$= \rho_r V_r L_r / \mu_r = 1 \qquad \ldots\ldots \quad (7)$$

where ρ_r = Mass density scale ratio
V_r = characteristic velocity scale ratio
L_r = Length scale ratio
μ_r = dynamic viscosity scale ratio

Equation (7) may be used to obtain the scale ratios for various other physical quantities on the basis of Reynolds model law.
Let us derive the scale ratios for models of certain quantities governed by Reynolds model law.

Scale ratio for Velocity (V_r):

From equation (7), $V_r = \mu_r / (\rho_r L_r)$ …… (8)

Scale ratio for time (T_r):

The scale ratio for velocity can be written as $V_r = V_m / V_p$

$$= (L_m / T_m) / (L_p / T_p)$$
$$= (L_m / L_p)(T_p / T_m)$$
$$= (L_m / L_p)\{1 / (T_m / T_p)\}$$
$$= L_r / T_r$$

Substituting $V_r = L_r / T_r$ in equation (7), we have,

$\{\rho_r (L_r / T_r) L_r\} / \mu_r = 1$
$\rightarrow \rho_r L_r^2 / \mu_r T_r = 1$
$\rightarrow T_r = \rho_r L_r^2 / \mu_r$ …… (9)

Scale ratio for acceleration (a_r):

Acceleration scale ratio, $a_r = a_m / a_p = \{L_m / (T_m)^2\} / \{L_p / (T_p)^2\}$

$$= (L_m / L_p)\{(T_p)^2 / (T_m)^2\}$$
$$= (L_m / L_p)[1 / \{(T_m)^2 / (T_p)^2\}]$$
$$= L_r / T_r^2$$

From equation (7), we have,
$V_r = \mu_r / (\rho_r L_r)$
$a_r = V_r / T_r = \{\mu_r / (\rho_r L_r)\} / \{\rho_r L_r^2 / \mu_r\}$

Putting the expression for T_r from equation (9) in the above expression, we have,

$a_r = \{\mu_r / (\rho_r L_r)\} / T_r = = \mu_r^2 / \rho_r^2 L_r^3$ …… (10)

<u>Scale ratio for discharge (Q_r):</u>

We know that, $Q_r = A_r V_r = L_r^2\ V_r$

From equation (7), we have,
$V_r = \mu_r / (\rho_r L_r)$

Hence, $Q = L_r^2\ \{\mu_r / (\rho_r L_r)\} = L_r \mu_r / \rho_r$ …… (11)

<u>Scale ratio for force (F_r):</u>

Shear force due to viscosity of fluid = shear stress x area
$= \tau A = \mu\ (V/y)\ A$
$F_r = \mu_r\ (V_r/y_r)A_r = \mu_r\ (V_r/L_r)\ (L_r^2) = \mu_r\ V_r\ L_r$

From equation (7), we have, $V_r = \mu_r / (\rho_r L_r)$

Putting $V_r = \mu_r / (\rho_r L_r)$ in the above expression for F_r

$F_r = \mu_r\ \{\mu_r / (\rho_r L_r)\}\ L_r = \mu_r^2 / \rho_r$ …… (12)

Some of the phenomena for which Reynolds model law can be a sufficient criterion for similarity of flow in the model and the prototype are:

(i) flow of incompressible fluid in closed pipes
(ii) motion of submarines completely under water
(iii) motion of air planes
(iv) flow around structures and other bodies immersed completely under moving fluids

(b) Froude Model Law

In case of flows where, in addition to the inertia force, the only other force of significance is the force of gravity, the similarity in flow in the model and the prototype can be obtained if the Froude number of flow is the same in both the model and the prototype. This is known as ***Froude Model Law***.

According to the law, we have,

$(Fr)_{model} = (Fr)_{prototype}$

$V_m / (L_m\ g_m)^{1/2} = V_p / (L_p\ g_p)^{1/2}$ ...... (13)

where $(Fr)_{model}$ = Froude number of flow in model
$(Fr)_{prototype}$ = Froude number of flow in prototype
V_m = velocity of flow in model
L_m = characteristic dimension (length) in model
g_m = acceleration due gravity at the site of model testing
V_p = velocity of flow in prototype
L_p = characteristic dimension (length) in prototype
g_p = acceleration due gravity at the site of prototype

Dividing LHS by RHS of above equation,

$V_m / (L_m g_m)^{1/2} / V_p / (L_p g_p)^{1/2} = 1$
$\rightarrow V_r / (g_r L_r)^{1/2} = 1$
$\rightarrow V_r = (g_r L_r)^{1/2}$ ...... (14)

Since in most cases, as the value of g at the site of model testing will practically be the same as the value of g at the site of the proposed prototype, we have the scale ratio of g, i.e., $g_r = g_m / g_p = 1$

Hence, equation (14) becomes

$V_r = L_r^{1/2}$
$\rightarrow V_r / L_r^{1/2} = 1$ …… (15)

Equation (14) or (15) may be used to obtain the scale ratios for various other physical quantities.

Let us derive the scale ratios for models of certain quantities governed by Froude model law.

Scale ratio for Time (T_r):

As discussed previously, the scale ratio for velocity can be written as
$V_r = L_r / T_r$

Substituting $V_r = L_r / T_r$ in equation (14), we have,

$L_r / T_r = (g_r L_r)^{1/2}$
$\rightarrow T_r = L_r / (g_r L_r)^{1/2} = L_r^{1/2} / g_r^{1/2}$ ...... (16)

Scale ratio for acceleration (a_r):

As discussed previously, acceleration scale ratio,

$a_r = L_r / T_r^2$

We have just expressed the scale ratio for time, T_r, as
$T_r = L_r^{1/2} / g_r^{1/2}$

Putting the above expression for T_r in $a_r = L_r / T_r^2$, we have,
$a_r = L_r / (L_r^{1/2} / g_r^{1/2})^2 = L_r / (L_r / g_r) = g_r$ ...... (17)

Scale ratio for discharge (Q_r):

We know that, $Q_r = A_r V_r = L_r^2 V_r$

From equation (14), we have, $V_r = (g_r L_r)^{1/2}$
Substituting the above expression for V_r in the expression for Q_r stated above, we have,

$Q_r = L_r^2 (g_r L_r)^{1/2} = L_r^{5/2} g_r^{1/2}$ …… (18)

Scale ratio for force (F_r):

Force due to gravity (weight of fluid) = mass x acceleration due to gravity
= Mg
where M = mass of fluid
g = acceleration due to gravity
M = mass density of fluid x volume of fluid = ρ x volume of fluid = ρL^3

Hence, $F = \rho L^3 g$

So, Scale ratio for force, $F_r = \rho_r L_r^3 g_r$ ...... (19)

Some of the phenomena for which Reynolds model law can be a sufficient criterion for dynamic similarity of flow in the model and the prototype are:

(i) Free-surface flows such as flow over spillways, sluices, etc.,
(ii) Flow of jet from an orifice or nozzle
(iii) Problems in which waves are likely to be formed on the surface
(iv) Problems in which fluids of different densities flow over one another

(c) Euler Model Law

In case of fluid systems where, in addition to the inertia force, the only other force of significance is the force due to supplied pressures, the dynamic similarity in flow in the model and the prototype can be obtained if the Euler number of flow is the same in both the model and the prototype. This is known as ***Euler Model Law***.

$(Eu)_{model} = (Eu)_{prototype}$

$[V_m / (p_m/\rho_m)^{1/2}] = [V_p / (p_p/\rho_p)^{1/2}]$ ...... (20)

where $(Eu)_{model}$ = Euler number of flow in model
$(Eu)_{prototype}$ = Euler number of flow in prototype
V_m = velocity of flow in the model
p_m = intensity of fluid pressure in the model
ρ_m = mass density of fluid in the model
V_p = velocity of flow in the prototype
p_p = intensity of fluid pressure in the prototype
ρ_p = mass density of fluid in the prototype

Dividing LHS by RHS of above equation,

$[V_m / (p_m/\rho_m)^{1/2}] / [V_p / (p_p/\rho_p)^{1/2}] = 1$

$\rightarrow [V_r / (p_r/\rho_r)^{1/2}] = 1$ ...... (21)

Equation (21) represents the Euler Model Law which may be used to evaluate scale ratios for various other physical quantities.

Euler model law may be considered as an essential requirement for establishing dynamic similarity in an enclosed fluid system where the turbulence is fully developed and the viscous forces are insignificant, and also the forces of gravity and surface tension are completely absent.

(d) Mach Model Law

In case of fluid flow phenomena where, in addition to the inertia force, the only other force of significance is the force resulting from elastic compression, the dynamic similarity in flow in the model and the prototype can be obtained if the Mach number of flow is the same in both the model and the prototype. This is known as ***Mach Model Law***.

$(Ma)_{model} = (Ma)_{prototype}$

$[V_m / (K_m / \rho_m)^{1/2}] = [V_p / (K_p / \rho_p)^{1/2}]$ ...... (22)

where $(Ma)_{model}$ = Mach number of flow in model

$(Ma)_{prototype}$ = Mach number of flow in prototype
V_m = velocity of flow in the model
K_m = bulk modulus of elasticity of fluid in the model
ρ_m = mass density of fluid in the model
V_p = velocity of flow in the prototype
K_p = bulk modulus of elasticity of fluid in the prototype
ρ_p = mass density of fluid in the prototype

Dividing LHS by RHS of above equation,

$$[V_m / (K_m / \rho_m)^{1/2}] / [V_p / (K_p / \rho_p)^{1/2}] = 1$$

$$\rightarrow [V_r / (K_r / \rho_r)^{1/2}] = 1 \qquad \ldots\ldots \qquad (23)$$

Equation (23) represents the Mach Model Law which may be used to evaluate scale ratios for various other physical quantities.

The Mach model law finds extensive application in aerodynamic testing and in phenomena involving velocities exceeding the speed of sound. It is also applicable in hydraulic model testing for cases of unsteady flow, especially water hammer problems.

(e) Weber Model Law

In case of fluid flow phenomena where, in addition to the inertia force, the only other force of significance is the force resulting from surface tension, the dynamic similarity in flow in the model and the prototype can be obtained if the Weber number of flow is the same in both the model and the prototype. This is known as ***Weber Model Law***.

$$(We)_{model} = (We)_{prototype}$$

$$[V_m / \{\sigma_m / (\rho_m L_m)\}^{1/2}] = [V_p / \{\sigma_p / (\rho_p L_p)\}^{1/2}] \qquad \ldots\ldots \qquad (24)$$

where $(We)_{model}$ = Weber number of flow in model
$(We)_{prototype}$ = Weber number of flow in prototype
V_m = velocity of flow in the model
σ_m = surface tension of fluid in the model
ρ_m = mass density of fluid in the model
L_m = characteristic length in the model
V_p = velocity of flow in the prototype
σ_p = surface tension of fluid in the prototype

ρ_p = mass density of fluid in the prototype
L_p = characteristic length in the prototype

Dividing LHS by RHS of above equation,

$[V_m / \{\sigma_m / (\rho_m L_m)\}^{1/2}] / [V_p / \{\sigma_p / (\rho_p L_p)\}^{1/2}] = 1$
$\rightarrow [V_r / \{\sigma_r / (\rho_r L_r)\}^{1/2}] = 1$ …… (25)

Equation (25) represents the Weber Model Law which may be used to evaluate scale ratios for various other physical quantities.

Weber model law can be applied in the following cases:

(i) flow over weirs involving very low heads
(ii) very thin sheet of liquid flowing over a surface
(iii) capillary waves in channels

TYPES OF MODELS

Hydraulic models can be broadly classified into two categories namely,

(i) Undistorted Models
(ii) Distorted Models

(i) Undistorted Models

An undistorted model is the one which is geometrically similar to its prototype, that is, the scale ratios for corresponding linear dimensions of the model and its prototype are same. As the basic condition of perfect similitude, i.e., geometric similarity, is satisfied, prediction in case of such models is relatively easy and many of the results obtained from the model tests can be transferred directly to the prototype.

(ii) Distorted Models

Distorted models are those in which one or more terms of the model are not identical with their counterparts in the prototype. As the basic condition of perfect similitude, i.e., geometric similarity, is not satisfied, the results obtained with the help of such models are liable to distortion and have more qualitative value only.

A distorted model may have either geometrical distortion, or distortion of hydraulic quantities or a combination of these.

What is geometric distortion?

The geometric distortion can either be *dimensional distortion* or *configurationally distortional*. For example, when the scale ratio adopted for the longitudinal dimension of the model and the prototype is different from the scale ratio adopted for the vertical dimension of the model and the prototype, the model is said to be *dimensionally distortional*. In general, when different scale ratios are adopted for the longitudinal, transverse and vertical dimensions, then it is said to be a distortion of dimensions.

Where dimensionally distorted models are frequently employed?

Distortion of dimensions is frequently adopted in river models where a different scale ratio is adopted for depth. In river models, the scales for vertical dimensions are larger than scales for horizontal dimensions. Such models are called '*vertically exaggerated models*'.

When the general configuration of the model does not bear a resemblance with its prototype, it results in a *configurationally distortional* model. For example, a river model will have a distortion of configuration if it is constructed with a bed-slope different from the one given by vertical exaggeration.

What is material distortion?

When the physical properties of the corresponding materials in the model and the prototype do not satisfy the similitude conditions, the material distortion arises.

Material distortion may have to be adopted in river models constructed for the studies of sediment transport.

Further, it may not be possible to obtain similitude in respect of certain uncontrollable hydraulic quantities such as time, discharge, etc., which may lead to distortion of hydraulic quantities.

Typical examples where distorted models are required:

(i) Rivers
(ii) Dams across very wide rivers
(iii) Harbours
(iv) Estuaries, etc.,

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In all the above cases, the horizontal dimensions are large in proportion to the vertical ones.

What are the reasons for adopting distorted models?

(i) to maintain accuracy in vertical measurements
(ii) to maintain turbulent flow
(iii) to obtain suitable bed material and its adequate movement
(iv) to obtain suitable roughness condition
(v) to accommodate the available facilities such as space, money, water supply and time.

Merits of Distorted Models

(i) the vertical exaggeration results in steeper water surface slopes and magnification of wave heights in models. Hence, the water surface slopes and the wave heights can be measured easily and accurately.
(ii) Due to exaggerated slopes, the Reynolds number of a model is considerably increased and the surface resistance is lowered. This assists in the simulation of flow conditions in the model and its prototype.
(iii) Sufficient tractive force can be developed to produce adequate bed movement with a reasonable small model.
(iv) Model size can be sufficiently reduced by distortion. This effects simplification in its operation and considerable reduction in cost.

Limitations of Distorted models ?

Example 6. A ship 150 m long moves in fresh water at 15°C at 36 km/h. A 1:100 model of this ship is to be tested in a towing basin containing a liquid of specific gravity 0.90. What viscosity must this liquid have for both Reynolds and Froude model laws to be satisfied? At what speed must the model be towed? If 117.7 watts is required to tow the model at this speed, what power is required by the ship? Dynamic viscosity of water at 15°C is 1.13×10^{-3} N.s/m^2.

Solution.

<u>Prototype Ship:</u>
Length of prototype ship, $L_p = 150\ m$
Velocity of prototype ship, $V_p = 36\ km/h = (36 \times 1000) / (1 \times 60 \times 60) = 10\ m/s$
Dynamic viscosity of water at 15°C, $\mu_p = 1.13 \times 10^{-3}\ N.s/m^2$
Mass density of water, $\rho_p = 1000\ kg\ (mass)/m^3$

Model Ship:

Length scale ratio = 1/100, i.e., $L_m / L_p = (1 / 100)$

Length of model ship, $L_m = (1 / 100) L_p = (1 / 100) (150) = 1.5$ m

Mass density of liquid, ρ_m = (specific gravity of liquid) x (mass density of water)

= 0.9 x 1000 $kg\ (mass)/m^3$ = 900 $kg\ (mass)/m^3$

Dynamic viscosity of liquid, μ_m = ?

Power required to tow the model, P_m = 117.7 watts

Reynolds model law and Froude model law are to be satisfied.

Reynolds model law:

$(N_R)_{model} = (N_R)_{prototype}$

$(\rho_m V_m L_m) / \mu_m = (\rho_p V_p L_p) / \mu_p$

Froude model law:

$(Fr)_{model} = (Fr)_{prototype}$

$V_m / (L_m g_m)^{1/2} = V_p / (L_p g_p)^{1/2}$

$$\rightarrow \frac{V_m}{[(1.5)(9.81)]^{1/2}} = \frac{10}{[(150)(9.81)]^{1/2}}$$

$\rightarrow V_m = 1$ m/s

Substituting $V_m = 1$ m/s in the Reynolds model law, we have,

$$\frac{(900)(1)(1.5)}{\mu_m} = \frac{(1000)(10)(150)}{1.13 \times 10^{-3}}$$

$\rightarrow \mu_m = 1.017 \times 10^{-6}$ N.s/m^2

Power required to tow the model ship, P_m = 117.7 watts.

We know that,

Power required to tow the model =

(total resistance experienced by the model ship) x (Velocity of model ship)

i.e., $P_m = R_m V_m$

where R_m = total resistance experienced by the model ship

$$R_m = \frac{P_m}{V_m} = \frac{117.7}{1} = 117.7\ N$$

The total resistance R experienced by a ship may be assumed to consist of two portions viz., (i) wave resistance R_w due to the action of waves; and (ii) the frictional resistance R_f due to frictional effects on the wetted surface of the ship.

That is,

$$R = R_w + R_f$$

Let the above equation for the prototype ship may be written as

$$R_p = (R_w)_p + (R_f)_p$$

and for the model ship as

$$R_m = (R_w)_m + (R_f)_m$$

The total resistance R encountered by a ship is a function of the velocity V of ship, viscosity μ of liquid (water) and mass density ρ of liquid (water), some characteristic length L to specify the size of the ship and the gravitational acceleration g. Hence, the functional relationship for R may be written as:

$R = \phi(V, \mu, \rho, L, g)$

This functional relationship can be expressed in terms of dimensionless parameters as

$$\frac{R}{\rho L^2 V^2} = \phi_1\left[\left(\frac{\rho VL}{\mu}\right), \left(\frac{V^2}{gL}\right)\right]$$

that is, $\dfrac{R}{\rho L^2 V^2} = \phi_1\left[N_R, (Fr)^2\right]$

Hence, for dynamic similarity between the model and the prototype for total resistance, we have,

$$\frac{R_p}{\rho_p L_p{}^2 V_p{}^2} = \frac{R_m}{\rho_m L_m{}^2 V_m{}^2}$$

$$\rightarrow \frac{R_p}{R_m} = \frac{\rho_p L_p{}^2 V_p{}^2}{\rho_m L_m{}^2 V_m{}^2} = \frac{(1000)(150)^2(10)^2}{(900)(1.5)^2(1)^2} = \mathbf{1111111.1}$$

$R_p = R_m$ x **1111111.1**= 130777777.7 N = 1.308 x 10^8 N

Hence, $P_p = R_p V_p$ = 1.308 x 10^8 x 10 = 1.307 x 10^9 Nm = 1.308 x 10^9 *watts*
= 1.308 x 10^6 kW

Example 7. A spillway 7.2 m high and 150 m long discharges 2150 m^3/s under a head of 4 m. If a 1:16 model of the spillway is to be constructed, find the model dimensions, head over the model and the model discharge.

Solution.

Prototype spillway:
Height of prototype spillway, $(Height)_p = 7.2\ m$
Length of prototype spillway, $L_p = 150\ m$
Discharge of prototype spillway, $Q_p = 2150\ m^3/s$
Head in prototype spillway, $(Head)_p = 4\ m$

Model spillway:
Length scale ratio, $L_r = L_m / L_p = 1 / 16$
Hence, length of model spillway, $L_m = (1 / 16)\ L_p = (1 / 16)\ (150) = 9.375\ m$
For geometric similarity of model and prototype,

$$\frac{(Height)_m}{(Height)_p} = \frac{L_m}{L_p}$$

$$\rightarrow (Height)_m = (Height)_p \frac{L_m}{L_p} = 7.2 \left(\frac{1}{16}\right) = 0.45\ m$$

$$\frac{(Head)_m}{(Head)_p} = \frac{L_m}{L_p}$$

$$(Head)_m = (Head)_p \frac{L_m}{L_p} = 4 \left(\frac{1}{16}\right) = 0.25\ m$$

The discharge equation for spillway is given by
$Q = C_d L H^{3/2}$
where C_d = coefficient of discharge of spillway
L = Length of spillway
H = Head of water in spillway

For kinematic similarity to exist between the model and the prototype, we have,

$$\frac{Q_m}{Q_p} = \frac{L_m H_m^{3/2}}{L_p H_p^{3/2}} = \frac{(9.375)(0.25)^{3/2}}{(150)(4)^{3/2}} = 9.765625 \times 10^{-4}$$

$Q_m = (9.765625 \times 10^{-4}) Q_p = (9.765625 \times 10^{-4})(2150) = 2.1\ m^3/s$

Example 8. In order to estimate the frictional head loss in a pipe 1 *m* in diameter, through which castor oil of specific gravity 0.96 and dynamic viscosity 9.9 *poise*, is to be transported at the rate of 5000 litres per second, a test was conducted on a pipe of diameter 50 *mm* using water at 15°*C* as the model fluid. Calculate the discharge required for the model pipe. If the head loss in 40 *m* length of the model pipe is measured as 13.6 *mm* of water, determine the corresponding head loss in the prototype. Also obtain the value of Darcy's friction factor for the prototype. Given absolute viscosity of water at 15°*C* = 0.0131 *poise*.

Solution.

Prototype pipe:

Diameter of pipe, $D_p = 1\ m$

Cross-sectional area of pipe, $A_p = \frac{\pi}{4} D_p^2 = \frac{\pi}{4}(1)^2 = 0.7854\ m^2$

Specific gravity of castor oil transported through pipe = 0.96

Mass density of castor oil, ρ_p = (specific gravity of castor oil) x (mass density of water)

$= 0.96 \times 1000 = 960\ kg\ (mass) / m^3$

Dynamic viscosity of castor oil, $\mu_p = 9.9\ poise = 9.9 \times 0.1\ N.s/m^2 = 0.99\ N.s/m^2$

Discharge rate of castor oil through the pipe, Q_p = 5000 litres per second

$= 5000 \times 10^{-3}\ m^3 / s$

$= 5\ m^3 / s$

Average velocity of flow through pipe, $V_p = Q_p / A_p = 5 / 0.7854 = 6.3662\ m/s$

Head loss due to friction in prototype pipe, $(H_f)_p$ = ?

Darcy's friction factor for the prototype pipe, f_p = ?

Model pipe:

Diameter of pipe, $D_m = 50\ mm = 50 \times 10^{-3}\ m = 0.05\ m$

Cross-sectional area of model pipe, $A_m = \frac{\pi}{4} D_m^2 = \frac{\pi}{4}(0.05)^2 = 1.963 \times 10^{-3}\ m^2$

Mass density of water, $\rho_m = 1000\ kg\ (mass) / m^3$

Discharge of water required for the model pipe, Q_m = ?

Length of model pipe, $L_m = 40\ m$

Head loss due to friction in 40 *m* length of model pipe, $(H_f)_m = 13.6\ mm$

$= 0.0136\ m$

Dynamic viscosity of water, $\mu_m = 0.0131\ poise = 0.0131 \times 0.1\ N.s/m^2$

$= 0.00131\ N.s/m^2$

For dynamic similarity of model and prototype, Reynolds model law must be applicable.

Reynolds model law:

$(N_R)_{model} = (N_R)_{prototype}$

$(\rho_m V_m L_m) / \mu_m = (\rho_p V_p L_p) / \mu_p$

Here, the characteristic dimension of pipe is its diameter; hence, $L_m = D_m$ and $L_p = D_p$ in the above expression

$$\rightarrow \frac{(960)(V_m)(0.05)}{0.00131} = \frac{(1000)(6.3662)(1)}{0.99}$$

$$\rightarrow V_m = 0.1755 \ m/s$$

Hence, discharge required for the model pipe, $Q_m = A_m \times V_m$

$= (1.963 \times 10^{-3}) \times (0.1755)$

$= 3.446 \times 10^{-4} \ m^3/s$

$= 0.345 \times 10^{-3} \ m^3/s$

= 0.345 litres per second

By dimensional analysis, the Resistance to flow is given by

$$R = \rho L^2 V^2 \phi\left(\frac{\rho V D}{\mu}\right)$$

As the Reynolds number for both the model and the prototype are the same, the value of the function ϕ in the above expression will be the same for both the model and the prototype. Hence, we can write,

$$\left(\frac{R}{\rho L^2 V^2}\right)_m = \left(\frac{R}{\rho L^2 V^2}\right)_p$$

Resistance to flow, R = (drop in pressure intensity due to head loss) x (cross-sectional area of pipe)

$$= (\rho g H_f)\left(\frac{\pi}{4} D^2\right)$$

Hence, we have,

$$\left[\frac{(\rho g H_f)\left(\frac{\pi}{4} D^2\right)}{\rho L^2 V^2}\right]_m = \left[\frac{(\rho g H_f)\left(\frac{\pi}{4} D^2\right)}{\rho L^2 V^2}\right]_p$$

$$\rightarrow \frac{(H_f)_m}{(H_f)_p} = \left(\frac{D_p{}^2}{D_m{}^2}\right)\left(\frac{L_m{}^2}{L_p{}^2}\right)\left(\frac{V_m{}^2}{V_p{}^2}\right)$$

As $\left(\frac{L_m}{L_p}\right) = \left(\frac{D_m}{D_p}\right) = \left(\frac{0.05}{1}\right)$

$$\frac{(H_f)_m}{(H_f)_p} = \left(\frac{1}{0.05}\right)^2\left(\frac{0.05}{1}\right)^2\left(\frac{0.1755}{6.3662}\right)^2 = 7.5997 \times 10^{-4}$$

It is given that $(H_f)_m = 0.0136 \ m$ of water in a length of 40 m

Hence, $(H_f)_p = (H_f)_m / (7.5997 \times 10^{-4}) = 0.0136 / (7.5997 \times 10^{-4})$

$= 17.896 \ m$ of castor oil

The corresponding length of prototype pipe in which this loss of 17.896 *m* occurs can be determined using the length scale ratio.

$$\frac{L_m}{L_p} = \frac{D_m}{D_p} = \frac{0.05}{1} = 0.05$$

$\rightarrow L_p = L_m / 0.05 = 40 / 0.05 = 800\ m$

Hence, loss of head per metre length of prototype pipe = $(H_f)_p / L_p$

= 17.896 / 800

= 0.02237 *m* of castor oil

= 22.37 *mm* of oil per *m* length of pipe

Darcy-Weisbach equation for head loss due to friction in pipe is given by

$$H_f = \frac{fL}{D}\frac{V^2}{2g}$$

Applying the above equation for the model pipe, we have,

$$\left(H_f\right)_p = \frac{f_p L_p}{D_p}\frac{{V_p}^2}{2g_p}$$

$$\rightarrow 17.896 = \frac{f_p (800)}{1}\frac{(6.3662)^2}{(2)(9.81)}$$

$\rightarrow f_p$ = friction factor for the prototype pipe = 0.01083