# FUNDAMENTALS OF FLUID FLOW

## INTRODUCTION

When a fluid is at rest, the only fluid property of significance is its specific weight. On the other hand, when a fluid is in motion, various other fluid properties also become significant. Therefore, the nature of flow of a real fluid is complex and cannot always be subjected to exact mathematical treatment. In such cases where exact mathematical analysis is not possible, one has to resort to experimentation. However, if some simplifying assumption could be made, the mathematical analysis of flow of fluids is possible.

### *What is Kinematics?*

It is the science which deals with the geometry of motion of fluids without considering the forces that cause the motion. It involves merely the description of motion of fluids in space – time relationship.

### *What is kinetics?*

It is the science which deals with the action of forces in causing the motion of fluids.

The study of flow of fluids involves both the kinematics and kinetics.

A fluid is composed of particles which move at different velocities and may be subjected to different accelerations. Further, even for a single fluid particle, the velocity and acceleration may change both with respect to space and time. Therefore, in the study of fluid flow, it becomes imperative to observe the motion of fluid particles at different points in space and at successive instants of time.

### *Methods of Description of Motion of Fluid*

1. *Lagrangian Method:* In this method, any individual fluid particle is selected and it is followed throughout its course of motion and observations are made regarding the behaviour of the particle during its course of motion through space.

2. *Eulerian Method:* Any point fixed in space that is occupied by flowing fluid particles is taken and observations are made with regard to the characteristics such as velocity, density and pressure of fluid at successive instants of time.

## VELOCITY OF FLUID PARTICLES

The motion of a fluid, like that of a solid, is described in terms of velocity. In case of solids, it is sufficient to measure the velocity of the body as a whole, as each and every particle composing the solid body moves with the same velocity as that of the whole body, whereas in case of a fluid, different fluid particles may move with different velocities at different points in space and at different points of time. *Therefore, how to define the velocity V, at any point, of a fluid particle?* The velocity of a fluid particle at a point can be defined as the ratio of the displacement of the fluid particle along its path of motion and the corresponding increment of time as the later approaches zero. Mathematically, it can be stated as:

$$\boldsymbol{V} = \lim_{dt \to 0} \frac{ds}{dt} \qquad \ldots\ldots (1)$$

where $V$ = velocity of fluid particle at a fixed point $P$ in space occupied by the fluid in motion
Let the coordinates of the point $P$ in space be $(x, y, z)$.
$ds$ = distance traversed by the fluid particle in the immediate vicinity of $P$
$dt$ = time taken by the fluid particle to traverse this distance $ds$

Figure 1 shows the path traced by a fluid particle in motion. The direction of the velocity vector $V$ at point $P$ is tangential to the path of fluid particle at $P$. The velocity vector $V$ has three components $u$, $v$ and $w$ in mutually perpendicular directions $x$, $y$ and $z$ respectively. The components of displacement $ds$ of the fluid particle along $x$, $y$ and $z$ directions are respectively $dx$, $dy$ and $dz$. Then, we have,

$$\boldsymbol{u} = \lim_{dt \to 0} \frac{dx}{dt}, \; \boldsymbol{v} = \lim_{dt \to 0} \frac{dy}{dt} \text{ and } \boldsymbol{w} = \lim_{dt \to 0} \frac{dz}{dt} \qquad \ldots\ldots (2)$$

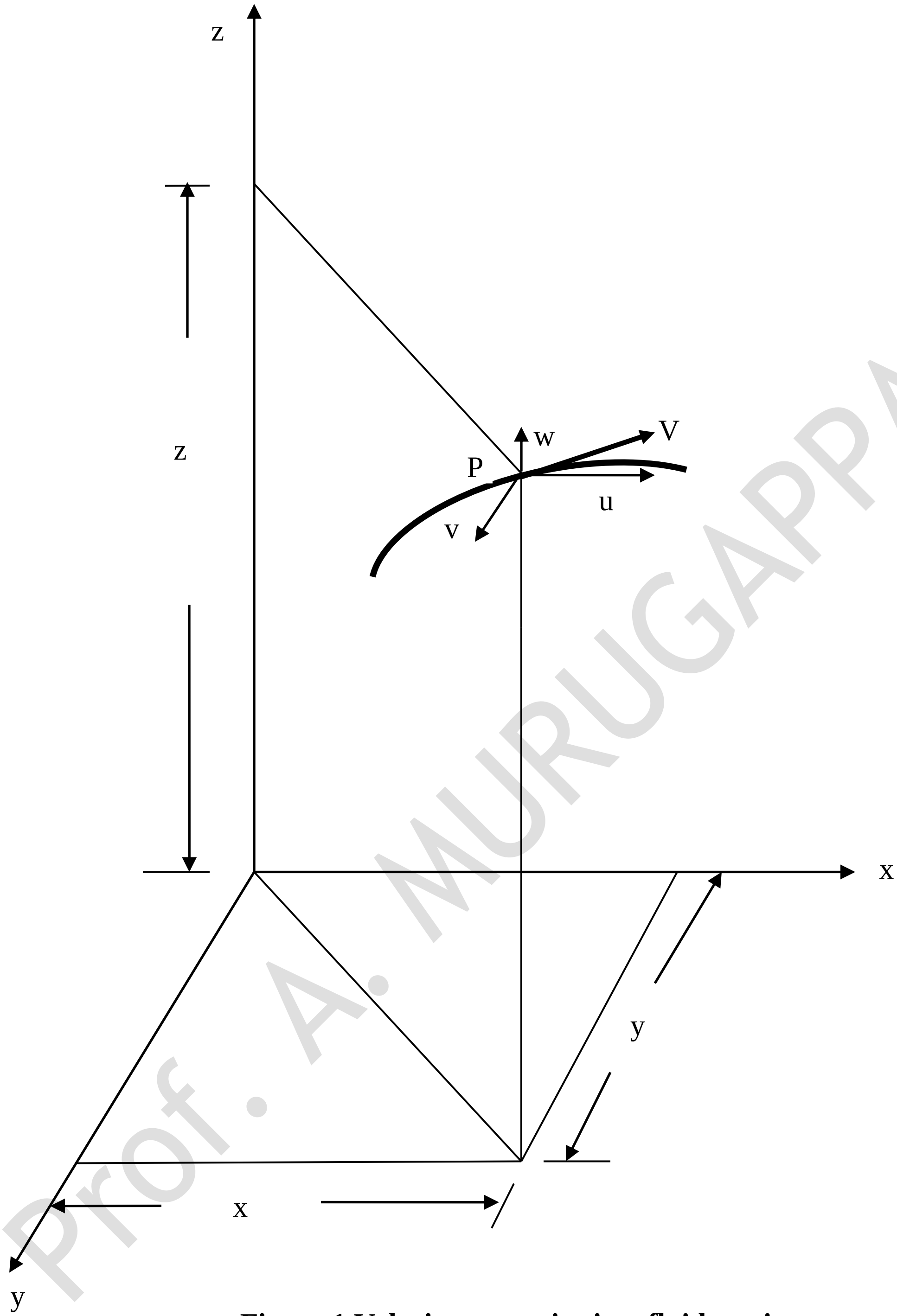



**Figure 1 Velocity at a point in a fluid motion**

The velocity $V$ of a fluid particle at any point is a function of space and time, that is, $V = f_1(x, y, z, t)$. Similarly, the velocity components $u$, $v$ and $w$ are also functions of space and time. That is, $u = f_2(x, y, z, t)$; $v = f_3(x, y, z, t)$ and $w = f_4(x, y, z, t)$.

In vector notation, the resultant velocity *V* may be expressed in terms of its components as

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w \qquad \text{...... (3)}$$

where, $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along *x*, *y* and *z* axes respectively.

## TYPES OF FLUID FLOW

**Steady flow:** Fluid flow is said to be steady, if at any point in the flowing fluid, the various characteristics such as velocity, pressure, density, temperature, etc., that describe the behaviour of fluid in motion, do not change with time. In other words, a flow is said to be steady, if the flow characteristics are independent of time. However, the flow characteristics may be different at different points in space. Mathematically, steady flow can be expressed as

$$\frac{\partial u}{\partial t} = 0; \frac{\partial v}{\partial t} = 0; \frac{\partial w}{\partial t} = 0; \frac{\partial \rho}{\partial t} = 0; \frac{\partial p}{\partial t} = 0 \qquad \text{...... (4)}$$

**Unsteady flow:** Fluid flow is said to be unsteady if at any point in the flowing fluid, one or more flow characteristics that describe the behaviour of fluid change with time. That is,

$$\frac{\partial V}{\partial t} \neq 0; \text{ and/or } \frac{\partial \rho}{\partial t} \neq 0 \text{ etc.,} \qquad \text{...... (5)}$$

**Uniform flow:** When the velocity of flow of fluid does not change, both in magnitude and direction, from point to point in the flowing fluid, at any given instant of time, the flow is said to be uniform. Mathematically, uniform flow can be stated as

$$\frac{\partial V}{\partial s} = 0 \qquad \text{...... (6)}$$

In the above expression, time is held constant; *s* represents any direction of displacement of the fluid elements.

**Non-uniform flow:** When the velocity of flow of fluid changes from point to point in the flowing fluid, at any given instant of time, the flow is said to be uniform. As velocity is a vector quantity, the change in velocity can occur due to change in both magnitude and direction, or due to change either in magnitude only or direction only. Mathematically, non-uniform flow can be stated as

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{s}} \neq 0 \qquad \text{...... (7)}$$

All the above four types of flows can exist independent of each other. The four types of combinations of above flows that are possible are:

(a) Steady – uniform flow
(b) Steady – non-uniform flow
(c) Unsteady – uniform flow
(d) Unsteady – non-uniform flow

Common example for each of the above combinations of flows:

(a) Flow of liquid through a long pipe line of constant diameter at constant discharge rate.
(b) Flow of liquid through a tapering pipe line (of either increasing or decreasing cross-sectional area) at constant discharge rate.
(c) Flow of liquid through long pipe line of constant diameter at either increasing or decreasing discharge rate
(d) Flow of liquid through a tapering pipe line at either increasing or decreasing discharge rate.

**Three – dimensional flow:** The different flow characteristics such as velocity, pressure, mass density and temperature are functions of space and time. That is, these characteristics may vary with the coordinates *x*, *y*, *z* of any point and time *t*. Such a flow is known as three – dimensional flow. If any of these flow characteristics does not change with respect to time, then it will be a steady three - dimensional flow.

**Two – dimensional flow:** When the flow characteristics of flowing fluid are functions of any two of the three co-ordinate directions and time *t*, that is, the flow characteristics are functions of, say, *x* and *y* directions and time *t*, then it is called a two –dimensional flow. In such a case, the flow characteristics do not change in *z* – direction. Further, if any of these flow characteristics does not change with time, it will be steady two-dimensional flow.

**One – dimensional flow:** When the flow characteristics of flowing fluid are functions of only any one of the three co-ordinate directions and time *t*, that is, the flow characteristics are functions of, say, *x* direction and time *t*, then it is called a one –dimensional flow. In such a case, the flow characteristics do not change in *y* and *z* directions. Further, if any of these flow characteristics does not change with time, it will be steady one-dimensional flow.

If we consider one of the characteristics of flow, say velocity of flow *V*, the following expressions may be written which clearly indicate the difference between the three-dimensional, two-dimensional and one-dimensional flows.

| Type of Flow | Unsteady | Steady |
|---|---|---|
| Three-dimensional | $V = f(x, y, z, t)$ | $V = f(x, y, z)$ |
| Two-dimensional | $V = f(x, y, t)$ | $V = f(x, y)$ |
| One-dimensional | $V = f(x, t)$ | $V = f(x)$ |

Similar expressions can be written for other characteristics of flowing fluid for the three types of flow mentioned above.

**Rotational flow:** When the fluid particles while moving in the direction of flow rotate about their mass centres, the flow is said to rotational.

**Irrotational flow:** When the fluid particles while moving in the direction of flow do not rotate about their mass centres, the flow is said to irrotational.

**Laminar flow:** When the various fluid particles move in layers (or laminae) with one layer of fluid sliding smoothly over an adjacent layer of fluid, the flow is said to be laminar. In the development of laminar flow, the viscosity of fluid plays a significant role. Hence, the flow of fluids that have high viscosity may be treated as laminar flow.

**Turbulent flow:** When the fluid particles move in an entirely haphazard or disorderly manner resulting in rapid and continuous mixing of the fluid particles leading to momentum transfer, the flow of fluid is said to be turbulent. In such a flow eddies or vortices of different shapes and sizes are present which move over long distances. The random movement of eddies give rise to fluctuations in velocity and pressure at any point in the flow field. These fluctuations in velocity and pressure are necessarily functions of time. Hence, at any point in the fluid flow, the velocity and pressure are functions of time thereby making such a flow unsteady. However, temporal mean values of velocity and pressure considered over sufficiently long time do not change with time.

It should be noted that the occurrence of turbulent flow is much more frequent than the occurrence of laminar flow conditions. Flow in natural streams, artificial channels, water supply pipes, sewers, etc., are a few examples of turbulent flow.

## DESCRIPTION OF THE FLOW PATTERN

The flow pattern of a flowing fluid may be described by the following:

1. Streamline
2. Stream-tube
3. Path-line
4. Streak-line

**1. Stream-line:** It is an imaginary curve drawn through a flowing fluid such that the tangent to the imaginary curve at any point gives the direction of the velocity of flow at that point. As a fluid is composed of fluid particles, the pattern of flow of fluid may be described a by a series of streamlines. This series of streamlines can be obtained by drawing a series of imaginary curves through the flowing fluid such that the velocity vector at any point on any curve is tangential to the curve.

Figure shows some of the streamlines for a flow pattern in $x$-$y$ plane. Let us consider a streamline passing through point $P$ whose co-ordinates are $x$ and $y$. The direction of velocity vector at point $P$ is tangential to the streamline. Let $u$ and $v$ be the components of velocity vector $V$, along $x$ and $y$ directions respectively. Let $dx$ and $dy$ be the components of the differential displacement $ds$ along the streamline in the immediate vicinity of $P$.

From figure, we have, $\tan\theta = \frac{v}{u} = \frac{dy}{dx}$ …… (8)

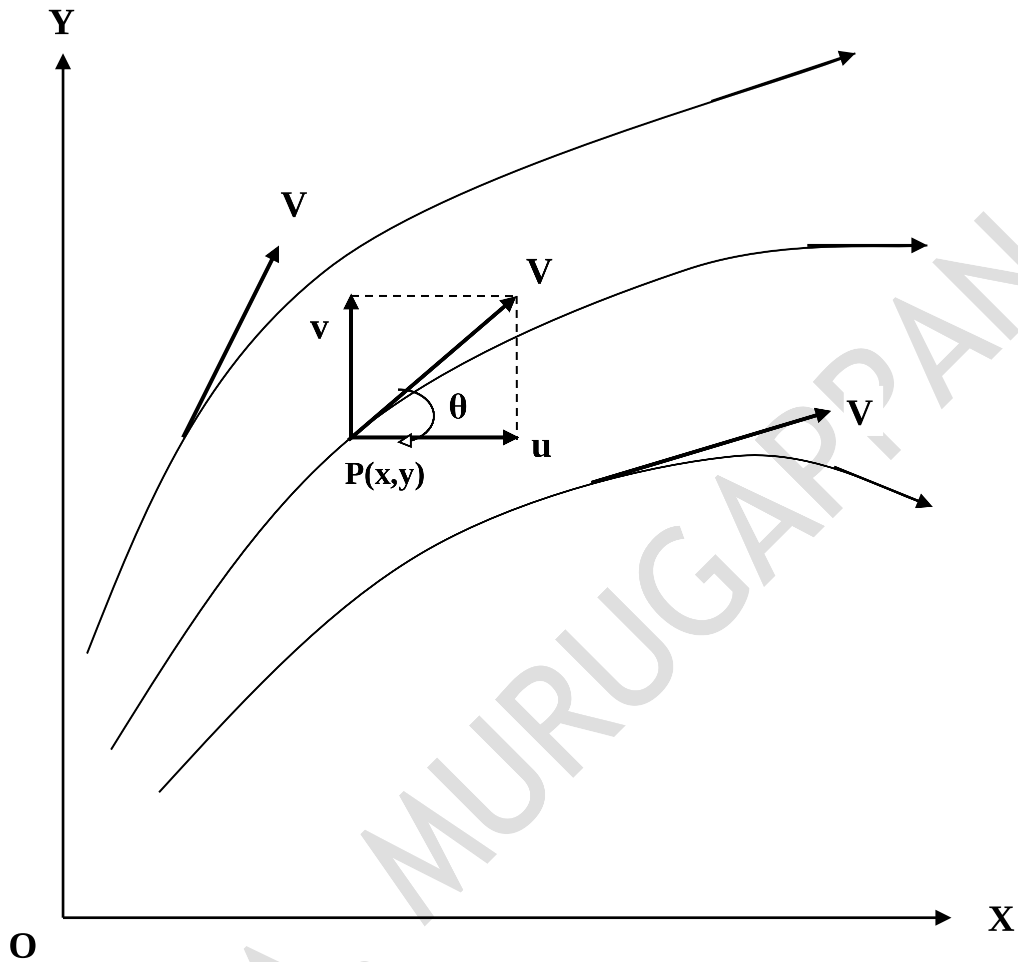



**Figure Streamlines for a flow pattern in x-y plane**



Therefore, we have,

$$\frac{dx}{u} = \frac{dy}{v} \qquad \ldots\ldots \quad (9)$$

i.e., $(u.dy - v.dx) = 0$

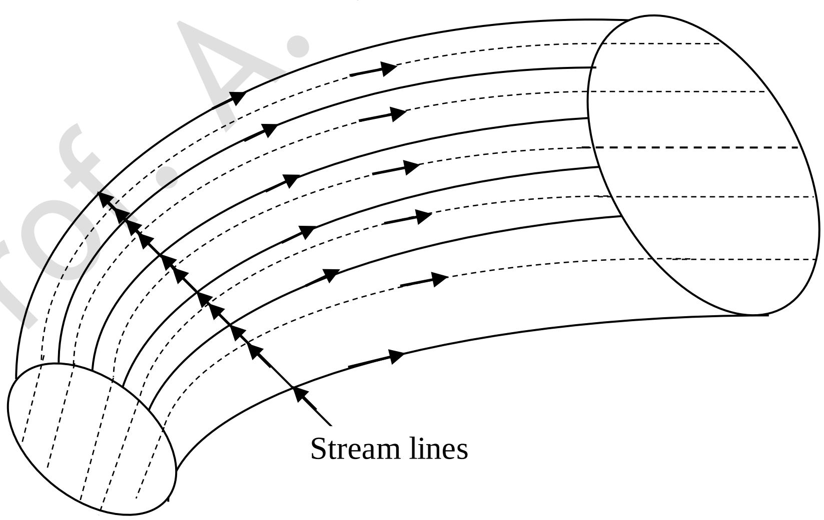
The differential equation for streamlines representing three dimensional flow can be obtained as

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \ldots\ldots \quad (10)$$

*Can there be flow of fluid across a streamline?* No . *Why?* We have already seen that the tangent to the streamline at any point gives the direction of the velocity of flow at that point. In other words, as a streamline is everywhere tangential to the velocity vector, there cannot be any component of the velocity at right angles to the streamline. Hence, there cannot be any flow across a streamline.

In case of steady flow, as the direction of velocity vector at any point does not change with time, the flow pattern does not change. Hence, the streamline pattern remains the same at different times for steady flow. In case of unsteady flow, as the direction of velocity vector at any point may change with time, the pattern of streamlines may also change from time to time.

**2. Stream tube.** It is an imaginary tube formed by a group of streamlines passing through a small closed curve, as shown in Figure. The stream tube is bounded on all sides by streamlines and as there is no component of velocity normal to any streamline, there cannot be any flow across the bounding surfaces. Hence, a fluid can enter or leave the stream tube only at its ends. If the cross-sectional area of the stream tube is small such that there is insignificant variation of velocity over it, the stream tube is called a stream filament.




**3. Path line**. It is defined as the imaginary line traced by a single fluid particle as it moves over a period of time. A path line indicates the direction of velocity vector of the same fluid particle at different instants of time. A streamline indicates the direction of velocity of a number of fluid particles at the same instant of time. A fluid particle always moves tangential to the streamline. In case of steady flow, streamlines are fixed in space. Therefore, in case of steady flow, both streamlines and path lines are identical.

## ACCELERATION OF A FLUID PARTICLE

*What is acceleration?*

It is defined as the time rate of change of velocity.

*Define acceleration of a fluid particle.*

The velocity of a fluid particle is a function of space (location of the point occupied by the fluid particle) and time. Let a fluid particle in space has a velocity *V*. Let the components of velocity *V* along the three mutually perpendicular directions namely, *x*, *y* and *z* directions be *u*, *v*, and *w* respectively. Then,

Acceleration of fluid particle in *x* – direction, $a_x = \lim_{dt \to 0} \frac{du}{dt}$

Acceleration of fluid particle in *y* – direction, $a_y = \lim_{dt \to 0} \frac{dv}{dt}$

Acceleration of fluid particle in *z* – direction, $a_z = \lim_{dt \to 0} \frac{dw}{dt}$

We have already stated that the velocity components *u*, *v* and *w* are all functions of the space co-ordinates *x*, *y* and *z* and time *t*. That is,

$$u = f(x,y,z,t)$$
$$v = f(x,y,z,t)$$
$$w = f(x,y,z,t)$$

The quantities $\frac{du}{dt}, \frac{dv}{dt}$ and $\frac{dw}{dt}$ represent respectively the total derivatives or substantial derivatives of the velocity components *u*, *v* and *w* with respect to time.

Let us consider the total derivative $\frac{du}{dt}$. As the velocity component *u* is a function of *x*, *y*, *z* and *t*, $\frac{du}{dt}$ can be expressed as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

Here, the partial derivative, $\frac{\partial u}{\partial x}$ represents the variation in velocity component *u* with respect to *x* – coordinate only. Similarly, the partial derivatives $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ represent the variation in velocity component u with respect to *y*- coordinate and *z* – coordinate respectively.

Also, it has been shown that,

$$\lim_{dt \to 0} \frac{dx}{dt} = u$$
$$\lim_{dt \to 0} \frac{dy}{dt} = v$$
$$\lim_{dt \to 0} \frac{dz}{dt} = w$$

Hence, taking limits on both sides of the expression for total derivative of velocity component *u* with respect to *t*, we have,

$$\lim_{dt \to 0} \frac{du}{dt} = \frac{\partial u}{\partial x} \lim_{dt \to 0} \frac{dx}{dt} + \frac{\partial u}{\partial y} \lim_{dt \to 0} \frac{dy}{dt} + \frac{\partial u}{\partial z} \lim_{dt \to 0} \frac{dz}{dt} + \frac{\partial u}{\partial t} \lim_{dt \to 0} \frac{dt}{dt}$$
$$\Rightarrow a_x = \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial z} w + \frac{\partial u}{\partial t}$$
$$\Rightarrow a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad \ldots\ldots \qquad (11)$$

By adopting the same procedure as above, the following expressions for other two components of acceleration namely, $a_y$ along *y* – direction and $a_z$ along *z* – direction are obtained.

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \qquad \ldots\ldots \qquad (12)$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \qquad \ldots\ldots \qquad (13)$$

Equations (11), (12) and (13) represent the expressions for the components of acceleration of a fluid particle in the three mutually perpendicular directions namely, *x*, *y* and *z*. In these expressions, the quantities $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ represent respectively the time rate of change of velocity components *u*, *v* and *w* with respect to time *t* at a particular point in the flow space. Hence, the quantities $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ are known as ***local accelerations*** or ***temporal*** (of time) ***accelerations***. The remaining quantities $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ represent the variation in velocity components of fluid particle due to change in position of fluid particle. Hence, these quantities are called ***convective accelerations***.

If the flow of a fluid particle is steady, then we have local acceleration equal to zero. That is, $\frac{\partial u}{\partial t} = 0; \frac{\partial v}{\partial t} = 0; \frac{\partial w}{\partial t} = 0$. When the flow of fluid article is steady, the convective acceleration need not be zero. Hence, the total acceleration or substantial acceleration need not be zero. However, in case the flow is uniform in addition to steady (steady-uniform flow), then the convective accelerations are also zero. Therefore, the total acceleration of fluid particle is also zero.

As velocity, acceleration is also a vector quantity. But, unlike velocity vector, the acceleration vector has no specific orientation with respect to the streamline, that is, acceleration vector need not be always tangential to the streamline as the velocity vector. In other words, acceleration vector may have any direction so that at any point it has components both tangential and normal to the streamline.

*When is a tangential acceleration developed?*

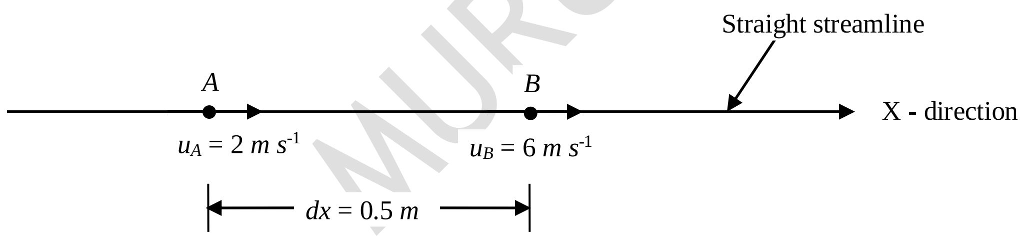
The tangential acceleration for a fluid particle is developed when *the magnitude of the velocity changes with respect to space and time*.

*When is a normal acceleration developed?*

The normal acceleration for a fluid particle is developed when the *fluid particle moves in a curved path along which the direction of velocity vector changes*. A normal acceleration may be due to the change in the direction of the velocity of the fluid particle irrespective of whether the magnitude of the velocity vector changes or not.

**Example 1.** In a steady flow two points *A* and *B* are 0.5 *m* apart on a straight streamline. If the velocity of flow varies linearly between *A* and *B* what is the acceleration at each point if the velocity at *A* is 2 $m\ s^{-1}$ and the velocity at *B* is 6 $m\ s^{-1}$.

**Solution**.




Let the straight streamline be oriented in *X* – direction. As the streamline is oriented in a unique direction (*X* – direction), the acceleration of the fluid flow is in only *X* – direction. There are no components of acceleration is the other two mutually perpendicular directions namely, *Y* – direction and *Z* – direction. In other words, the fluid flow has only acceleration component along *X* – direction and no acceleration components in the other two directions. That is, the fluid flow has $a_x$, while $a_y$ and $a_z$ are zero.

The general form of expression for $a_x$ is given by

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

When the flow is steady, the velocity component *u* does not undergo any change with respect to time. Hence, the above expression reduces to

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Further for steady flow with straight streamlines (oriented in *X* – direction), there is no change in velocity component *u* in *Y* and *Z* directions. Hence, the above equation reduces to

$$a_x = u\frac{\partial u}{\partial x}$$

Here, $u_A$ = velocity of flow at point *A* (in *X* – direction) = 2 $m\ s^{-1}$
$u_B$ = velocity of flow at point *B* (in *X* – direction) = 6 $m\ s^{-1}$
$\left(\frac{\partial u}{\partial x}\right)_A$ = change in velocity of flow at point *A* in *X* – direction with respect to distance along *X* – direction

As the velocity of flow varies linearly between *A* and *B*,

$$\left(\frac{\partial u}{\partial x}\right)_A = \frac{u_B - u_A}{dx} = \frac{6-2}{0.5} = 8\ s^{-1}$$

Hence, acceleration of flow at point *A* in *X* – direction is given by

$$(a_x)_A = u_A\left(\frac{\partial u}{\partial x}\right)_A = \left(2\ m\ s^{-1}\right) \text{x} \left(8\ s^{-1}\right) = 16\ m\ s^{-2}$$

Let $\left(\frac{\partial u}{\partial x}\right)_A$ = change in velocity of flow at point *B* in *X* – direction with respect to distance along *X* – direction

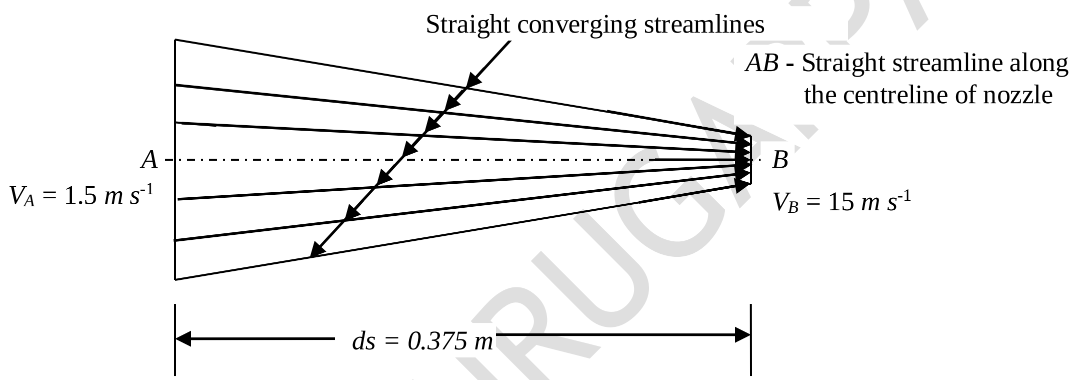
$$\left(\frac{\partial u}{\partial x}\right)_B = \frac{u_B - u_A}{dx} = \frac{6-2}{0.5} = 8\ s^{-1}$$

Hence, acceleration of flow at point *B* in *X* – direction is given by

$$(a_x)_B = u_B\left(\frac{\partial u}{\partial x}\right)_B = \left(6\,m\,s^{-1}\right)\text{x}\left(8\,s^{-1}\right) = 48\,m\,s^{-2}$$

**Example 2.** A nozzle is so shaped that the velocity of flow along the centerline changes linearly from 1.5 $m\ s^{-1}$ to 15 $m\ s^{-1}$ in a distance of 0.375 $m$. Determine the magnitude of the convective acceleration at the beginning and end of this distance.

**Solution**.




In a steady flow when the streamlines are straight and converging then there will only be convective tangential acceleration, $a_s$. The convective normal acceleration, $a_n$, is zero. The expression for $a_s$ is given by

$$a_s = V_s\frac{\partial V_s}{\partial s}$$

Let $(a_s)_A$ = convective tangential acceleration at $A$

$(a_s)_B$ = convective tangential acceleration at $B$

$(V_s)_A$ = velocity component along tangential direction = $V_A$ = 1.5 $m$ s$^{-1}$

$(V_s)_B$ = velocity component along tangential direction = $V_B$ = 15 $m$ s$^{-1}$

$\left(\frac{\partial V_s}{\partial s}\right)$ = change in velocity component $V_s$ due to change in the position of the fluid particle

$$= \frac{(V_s)_B - (V_s)_A}{ds} = \frac{15-1.5}{0.375} = 36\,m\,s^{-1}\text{ per }m$$

$(a_s)_A = (V_s)_A \left(\frac{\partial V_s}{\partial s}\right) =$ (1.5 $m\ s^{-1}$) x (36 $m\ s^{-1}$ per $m$) = 54 $m\ s^{-2}$

$(a_s)_B = (V_s)_B \left(\frac{\partial V_s}{\partial s}\right) =$(15 $m\ s^{-1}$) x (36 $m\ s^{-1}$ per $m$) = 540 $m\ s^{-2}$

## CONTINUITY EQUATION

The continuity equation represents the mathematical statement of the principle of conservation of mass. Let us consider a fixed region in space within a flowing fluid. As per the principle of conservation of mass, fluid can neither be created nor be destroyed within this fixed region in space. Hence, the time rate of increase of mass of fluid within the fixed region must be equal to the difference between the rate at which fluid mass enters the fixed region and the rate at which the fluid mass leaves the fixed region. If the flow is steady, the time rate of increase of mass of fluid within the fixed region is zero and the rate at which the fluid mass enters the fixed region is equal to the rate at which the fluid mass leaves the fixed region.

Let us consider an elementary rectangular parallelepiped with sides of length $\delta x$, $\delta y$ and $\delta z$ as shown in Figure below. Let $O$ denote the centre of the parallelepiped. Let the coordinates of the point $O$ be ($x$, $y$, $z$). Let the components of velocity of fluid at point $O$ in $x$, $y$ and $z$ directions be respectively $u$, $v$ and $w$. Let $\rho$ be the mass density of fluid. Let the face *PQRS* oriented perpendicular to the $x$ – direction passes through the centre $O$ of the parallelepiped.

Mass of fluid passing per unit time through the central face *PQRS*, in $x$ - direction = (mass density of fluid) x (component of velocity in x – direction) x (area of central face *PQRS* defined perpendicular to the $x$ – direction)
= $\rho\, u\, (\delta y\ \delta z)$

Let us consider that the mass of fluid flowing through the parallelepiped per unit time varies in $x$ – direction.

Hence, mass of fluid flowing per unit time through the left face *ABCD* of the parallelepiped, in $x$ - direction = mass of fluid flowing per unit time through the central face *PQRS*, in $x$ – direction + variation in mass of fluid flowing per unit time, in $x$ – direction, in a length equal to $\frac{\delta x}{2}$ between the left face *ABCD* and the central face *PQRS*

$$= (\rho u \delta y \delta z) + \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(-\frac{\delta x}{2}\right)$$

In the above expression, the negative sign prefixing the length $\frac{\delta x}{2}$ indicates that the face *ABCD* is situated to the left of the central face *PQRS* in which the centre *O* of the parallelepiped lies.

Similarly, mass of fluid flowing per unit time through the right face *EFGH* of the parallelepiped, in $x$ – direction = mass of fluid flowing per unit time through the central face *PQRS,* in $x$ – direction + variation in mass of fluid flowing per unit time, in $x$ – direction, in a length equal to $\frac{\delta x}{2}$ between the central face *PQRS* and the right face *EFGH*

$$= (\rho u \delta y \delta z) + \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(\frac{\delta x}{2}\right)$$

Net mass of fluid that has remained in the parallelepiped per unit time between the pair of faces *ABCD* and *EFGH* =
(Mass of fluid flowing per unit time through the left face *ABCD* of the parallelepiped, in $x$ – direction) – (Mass of fluid flowing per unit time through the right face *EFGH* of the parallelepiped, in $x$ – direction) =

$$\left[(\rho u \delta y \delta z) + \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(-\frac{\delta x}{2}\right)\right] - \left[(\rho u \delta y \delta z) + \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(\frac{\delta x}{2}\right)\right]$$

$$= (\rho u \delta y \delta z) - \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(\frac{\delta x}{2}\right) - (\rho u \delta y \delta z) - \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(\frac{\delta x}{2}\right)$$

$$= -2\ \frac{\partial}{\partial x}(\rho u \delta y \delta z)\left(\frac{\delta x}{2}\right)$$

$$= -\frac{\partial}{\partial x}(\rho u \delta y \delta z)\delta x$$

$$= -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z \qquad \ldots\ldots \quad (14)$$

The area $\delta y \delta z$ is taken out of the parentheses since it is not a function of $x$. The quantity ($\delta x \delta y \delta z$) represents the volume of the parallelepiped.

In a similar manner, one can obtain the values of net mass of fluid that has remained in the parallelepiped per unit time through the other two pairs of faces of the parallelepiped.

Net mass of fluid that has remained in the parallelepiped per unit time between the pair of faces *ABFE* and *DCGH* =
(Mass of fluid flowing per unit time through the front face *ABFE* of the parallelepiped, in *y* – direction) – (Mass of fluid flowing per unit time through the rear face *DCGH* of the parallelepiped, in *y* – direction) =

$$\left[(\rho v\delta x\delta z)+\frac{\partial}{\partial y}(\rho v\delta x\delta z)\left(-\frac{\delta y}{2}\right)\right] - \left[(\rho v\delta x\delta z)+\frac{\partial}{\partial y}(\rho v\delta x\delta z)\left(\frac{\delta y}{2}\right)\right]$$

$$= (\rho v\delta x\delta z)-\frac{\partial}{\partial y}(\rho v\delta x\delta z)\left(\frac{\delta y}{2}\right) - (\rho v\delta x\delta z)-\frac{\partial}{\partial y}(\rho v\delta x\delta z)\left(\frac{\delta y}{2}\right)$$

$$= -2\ \frac{\partial}{\partial y}(\rho v\delta x\delta z)\left(\frac{\delta y}{2}\right)$$

$$= -\frac{\partial}{\partial y}(\rho v\delta x\delta z)\delta y$$

$$= -\frac{\partial}{\partial y}(\rho v)\delta x\delta y\delta z \qquad \ldots\ldots\ (15)$$

The area $\delta x\delta z$ is taken out of the parentheses since it is not a function of *y*.

Net mass of fluid that has remained in the parallelepiped per unit time between the pair of faces *ADHE* and *BCGF* =
(Mass of fluid flowing per unit time through the bottom face *ADHE* of the parallelepiped, in *z* – direction) – (Mass of fluid flowing per unit time through the top face *BCGF* of the parallelepiped, in *z* – direction) =

$$\left[(\rho w\delta x\delta y)+\frac{\partial}{\partial z}(\rho w\delta x\delta y)\left(-\frac{\delta z}{2}\right)\right] - \left[(\rho w\delta x\delta y)+\frac{\partial}{\partial z}(\rho w\delta x\delta y)\left(\frac{\delta z}{2}\right)\right]$$

$$= (\rho w\delta x\delta y)-\frac{\partial}{\partial z}(\rho w\delta x\delta y)\left(\frac{\delta z}{2}\right) - (\rho w\delta x\delta y)-\frac{\partial}{\partial z}(\rho w\delta x\delta y)\left(\frac{\delta z}{2}\right)$$

$$= -2\ \frac{\partial}{\partial z}(\rho w\delta x\delta y)\left(\frac{\delta z}{2}\right)$$

$$= -\frac{\partial}{\partial z}(\rho w\delta x\delta y)\delta z$$

$$= -\frac{\partial}{\partial z}(\rho w)\delta x\delta y\delta z \qquad \ldots\ldots\ (16)$$

The area $\delta x \delta y$ is taken out of the parentheses since it is not a function of *z*.

The net total mass of fluid that has remained in the parallelepiped per unit time can be obtained by summing up the net mass of fluid that has remained in the parallelepiped between the pairs of faces *ABCD* and *EFGH*, *ABFE* and *DCGH*, and *ADHE* and *BCGF*.

Net total mass of fluid that has remained in the parallelepiped per unit time =

(14) + (15) + (16) = $\left[-\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z\right] + \left[-\frac{\partial}{\partial y}(\rho v)\delta x \delta y \delta z\right] + \left[-\frac{\partial}{\partial z}(\rho w)\delta x \delta y \delta z\right]$ …… (17)

Mass of fluid in the parallelepiped = (Mass density of fluid) x (Volume of parallelepiped)

$= \rho(\delta x \delta y \delta z)$

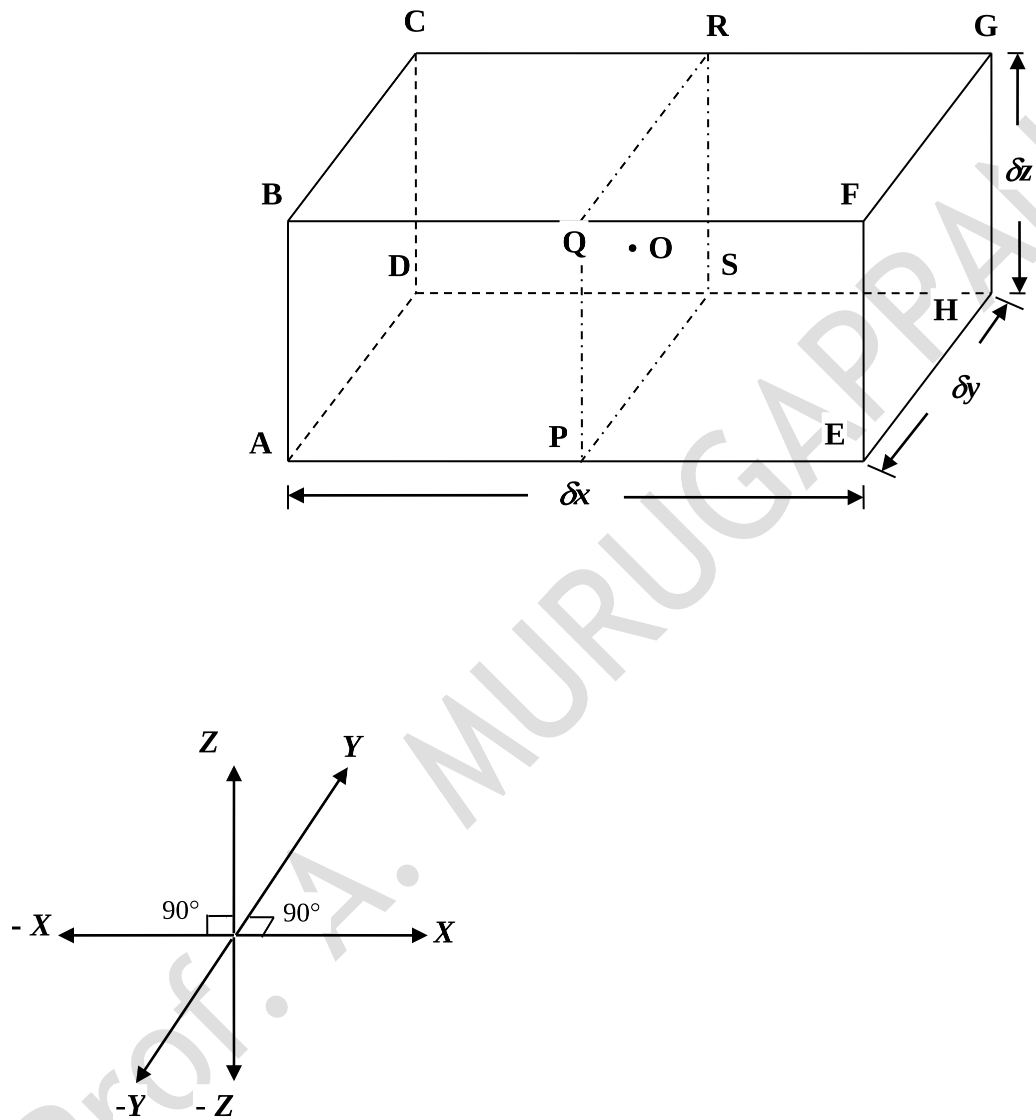



**Figure: Fluid element with forces acting on it in a static mass of fluid**

Time rate of change of mass of fluid in the parallelepiped = $\frac{\partial}{\partial t}\{\rho(\delta x\delta y\delta z)\}$

…… (18)

As per the law of conservation of mass, the net total mass of fluid that has remained in the parallelepiped per unit time (given by Eq. (17)) must equal the time rate of change of mass of fluid in the parallelepiped (given by Eq. (18))

$$\left[-\frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z\right] + \left[-\frac{\partial}{\partial y}(\rho v)\delta x\delta y\delta z\right] + \left[-\frac{\partial}{\partial z}(\rho w)\delta x\delta y\delta z\right] = \frac{\partial}{\partial t}\{\rho(\delta x\delta y\delta z)\}$$

$$\Rightarrow \frac{\partial}{\partial t}\{\rho(\delta x\delta y\delta z)\} + \frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z + \frac{\partial}{\partial y}(\rho v)\delta x\delta y\delta z + \frac{\partial}{\partial z}(\rho w)\delta x\delta y\delta z = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t}(\delta x\delta y\delta z) + \frac{\partial(\rho u)}{\partial x}(\delta x\delta y\delta z) + \frac{\partial(\rho v)}{\partial y}(\delta x\delta y\delta z) + \frac{\partial(\rho w)}{\partial z}(\delta x\delta y\delta z) = 0$$

Dividing the expression by the volume of the parallelepiped, $(\delta x\delta y\delta z)$ ,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \ldots\ldots \ (19)$$

Equation (19) gives the continuity equation in its most generalized form, applicable to steady or unsteady flow, uniform or non-uniform flow, in three dimensions, of both compressible as well as incompressible fluid.

If the flow is steady, $\frac{\partial \rho}{\partial t} = 0$, hence, Eq. (6) reduces to the form,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \ldots\ldots \ (20)$$

Further, if the flowing fluid is incompressible, Eq. (20) reduces to the form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \ldots\ldots \ (21)$$

For flow of fluid (compressible or incompressible) in two-dimensions (2-*D* flow), the continuity equation in its most generalized form, applicable to steady or unsteady flow, uniform or non-uniform flow, is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \text{……} \quad (22)$$

In writing Eq. (22), it is assumed that the flow characteristics do not vary in the *z* – direction and they vary only in the other two mutually perpendicular reference directions *x* and *y*.

For steady two-dimensional flow of a compressible fluid, Eq. (22) reduces to the form,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \text{……} \quad (23)$$

For steady two-dimensional flow of an incompressible fluid, Eq. (23) reduces to the form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{……} \quad (24)$$

The continuity equation for one-dimensional flow, in its most generalized form, applicable for steady or unsteady flow, uniform or non-uniform flow, of both compressible and incompressible fluid can be written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \qquad \text{……} \quad (25)$$

In writing Eq. (25), it is assumed that the flow characteristics are functions of only *x* – direction and they do not vary in the other two mutually perpendicular reference directions, *y* and *z*.

Further, if the flow is steady, Eq. (25) reduces to the form,

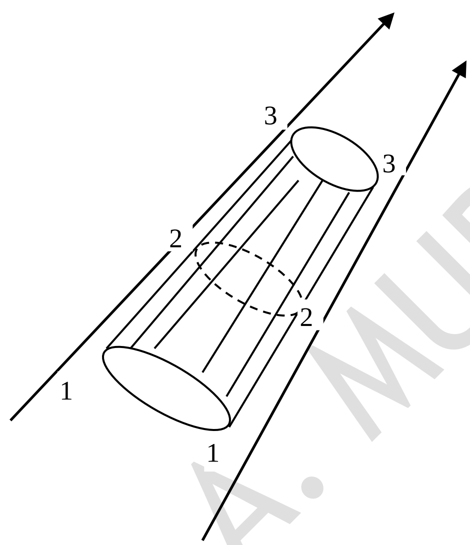
$$\frac{\partial(\rho u)}{\partial x} = 0 \qquad \text{……} \quad (26)$$

Further, if the flowing fluid is incompressible, Eq. (26) reduces to the form,

$$\frac{\partial u}{\partial x} = 0 \qquad \text{……} \quad (27)$$

The continuity equation in the above form does not contain the area of flow passage. Hence, the continuity equation derived is applicable in case of flow in which the flow passage has a uniform cross-sectional area. In fact, a true one-dimensional flow would occur only in case of a straight flow passage that has uniform cross-sectional area. However, one-dimensional flow may also be assumed to occur in case of straight or curved flow passage with varying cross-sectional area provided the velocity of flow is uniform at each section of flow passage.

**Continuity Equation for One-dimensional Flow Containing Variation of Cross-sectional Area of the Flow Passage**




$\rho A V = \text{constant}$ …… (28)

$\Rightarrow \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \rho_3 A_3 V_3 = \ldots\ldots = \text{constant}$ ……(28a)

where $\rho_1$, $\rho_2$, $\rho_3$, ……, represent mass density of fluid at cross sections 1, 2, 3, …… respectively. $A_1$, $A_2$, $A_3$, ……, represent area of cross-section of flow passage at sections 1, 2, 3, ……, respectively. $V_1$, $V_2$, $V_3$, ……, represent mean velocity of flow at sections 1, 2, 3, ……, respectively. Equation (28a) represents thus represents the continuity equation applicable for a steady one-dimensional flow of compressible as well as incompressible fluids.

When the flowing fluid is incompressible, the mass density of fluid does not change with time and space. That is, mass density of fluid remains a constant. Hence, equation (28a) becomes

$A_1V_1 = A_2V_2 = A_3V_3 = \ldots\ldots = \text{constant}$ ...... (28b)

where $A_1$, $A_2$, $A_3$, ....., and $V_1$, $V_2$, $V_3$ are as defined earlier.

The product of area of cross section of flow passage, $A$, and mean velocity of flow, $V$, gives the discharge or the volume of fluid flowing per unit time through any section. The SI unit of discharge is $m^3$/s. Therefore, equation (18a) can be written as

$Q = A_1V_1 = A_2V_2 = A_3V_3 = \ldots\ldots = \text{constant}$ ...... (28c)

Equation (28c) represents the equation of continuity applicable to a steady one-dimensional flow of incompressible fluid.

**Example 3.** Determine which of the following pairs of velocity components $u$ and $v$ satisfy the continuity equation for a two-dimensional flow of an incompressible fluid.

(a) $u = Cx; v = -Cy$
(b) $u = (3x - y); v = (2x + 3y)$
© $u = (x + y); v = (x^2 - y)$
(d) $u = A\sin xy; v = -A\sin xy$
(e) $u = 2x^2 + 3y^2; v = -3xy$

**Solution.**

The continuity equation in differential form for a steady two-dimensional flow of an incompressible fluid can be written as $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Here, it is assumed that the flow characteristics vary only in the two co-ordinate directions x and y.

(a) $u = Cx; v = -Cy$

$\frac{\partial u}{\partial x} = C$ ; $\frac{\partial v}{\partial y} = -C$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = C + (-C) = 0$

So, the given pair of velocity components *u* and *v* satisfies the continuity equation for two-dimensional flow of an incompressible fluid.

(b) $u = (3x - y); v = (2x + 3y)$

$\frac{\partial u}{\partial x} = 3$; $\frac{\partial v}{\partial y} = 3$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 3 + 3 = 6 \neq 0$

Hence the velocity components *u* and *v* do not satisfy the continuity equation for two-dimensional flow of an incompressible fluid.

© $u = (x + y); v = (x^2 - y)$

$\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial y} = -1$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 + (-1) = 0$

So, the given pair of velocity components *u* and *v* satisfies the continuity equation for two-dimensional flow of an incompressible fluid.

(d) $u = A \sin xy; v = -A \sin xy$

$\frac{\partial u}{\partial x} = A(\cos xy) y = Ay(\cos xy)$

$\frac{\partial v}{\partial y} = -A \cos(xy) x = -Ax(\cos xy)$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Ay(\cos xy) - Ax(\cos xy) \neq 0$

(e) $u = 2x^2 + 3y^2; v = -3xy$

$\frac{\partial u}{\partial x} = 2(2x) = 4x$

$\frac{\partial v}{\partial y} = -3x$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4x + (-3x) = x$

**Example 4.** Calculate the unknown velocity components so that they satisfy the continuity equation for three-dimensional flow of an incompressible fluid

(a) $u = 2x^2; v = xyz; w = ?$
(b) $u = (2x^2 + 2xy); w = (z^3 - 4xz - 2yz); v = ?$

**Solution.**

The continuity equation in its differential form applicable to three – dimensional flow of an incompressible fluid is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(a) $u = 2x^2; v = xyz; w = ?$

$$\frac{\partial u}{\partial x} = 2(2x) = 4x; \frac{\partial v}{\partial y} = xz(1) = xz$$

Substituting the values of $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$ and $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}}$ in the continuity equation mentioned above, we have,

$$4x + xz + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w}{\partial z} = -4x - xz$$

$$\Rightarrow w = \int(-4x - xz)dz = -4xz - xz^2 + f(x, y)$$

(b) $u = (2x^2 + 2xy); w = (z^3 - 4xz - 2yz); v = ?$

$$\frac{\partial u}{\partial x} = 2(2x) + 2y(1) = 4x + 2y; \frac{\partial w}{\partial z} = 3z^2 - 4x(1) - 2y(1) = 3z^2 - 4x - 2y$$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial w}{\partial z}$ in the continuity equation mentioned above, we have,

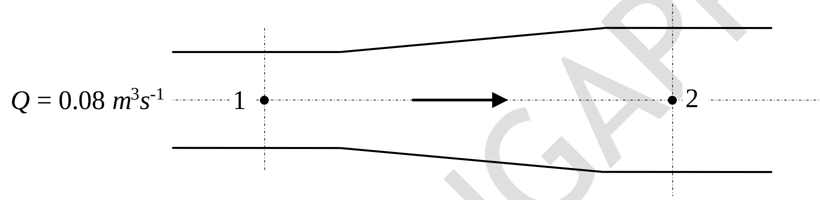
$$(4x+2y)+\frac{\partial v}{\partial y}+\ 3z^2-4x-2y\ =0$$

$$\Rightarrow\ \frac{\partial v}{\partial y}=-4x-2y-3z^2+4x+2y=3z^2$$

$$\Rightarrow\ v=\int 3z^2.dy=3z^2(y)+f(x,z)=3yz^2+f(x,z)$$

**Example 5**. A pipe of diameter 0.2 *m* increases gradually to 0.3 *m*. If it carries 0.08 $m^3\ s^{-1}$ of water, what are the velocities at the two sections?

**Solution.**




Diameter of pipe at section 1, $D_1$ = 0.2 *m*

Area of cross-section of pipe at section 1, $A_1 = \frac{\pi}{4}D_1^2=\frac{\pi}{4}(0.2)^2=0.0314\ m^2$

Diameter of pipe at section 2, $D_2$ = 0.3 *m*

Area of cross-section of pipe at section 2, $A_2 = \frac{\pi}{4}D_2^2=\frac{\pi}{4}(0.3)^2=0.0707\ m^2$

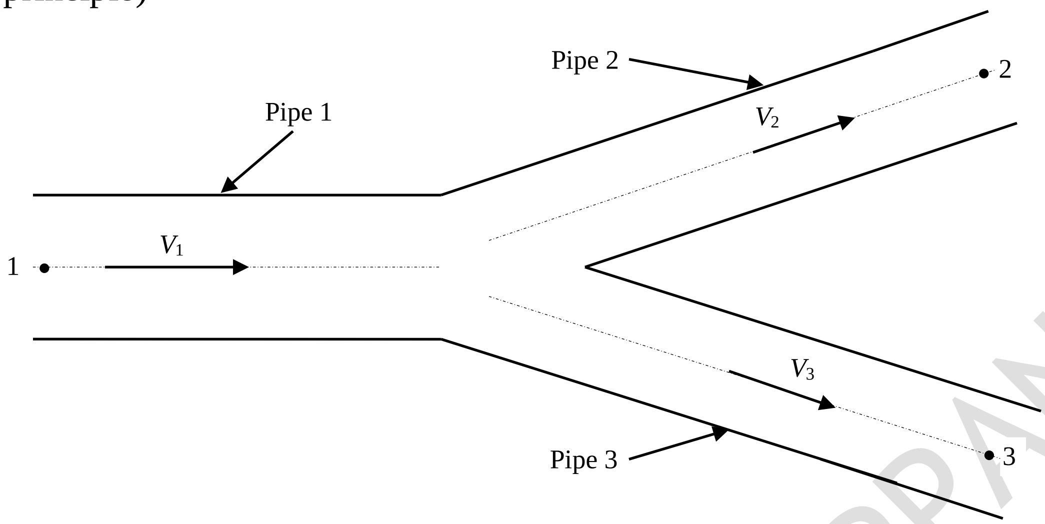
Applying equation of continuity between sections 1 and 2

$$Q=A_1V_1=A_2V_2$$

$$\Rightarrow 0.08\ m^3s^{-1} = (0.0314\ m^2)V_1 = (0.0707\ m^2)V_2$$

$$\Rightarrow V_1 = \frac{0.08}{0.0314}=2.548\ \text{ms}^{-1}\ \text{and}\ V_2 = \frac{0.08}{0.0707}=1.131\ \text{ms}^{-1}$$

**Example 6**. Water flows through a branching pipeline as shown in the figure. If the diameter, $D_2$, is 250 *mm*, $V_2$ = 1.77 $m\ s^{-1}$ and $V_3$ = 1.43 $m\ s^{-1}$, find (a) diameter, $D_3$, required for $Q_3 = 2Q_2$ and (b) the total discharge at section 1.




Pipe 1 of diameter $D_1$ branches into two pipes namely, pipe 2 of diameter $D_2$ and pipe 3 of diameter $D_3$. Let the mean velocity of flow in pipe 1 be $V_1$. The mean velocities of flow in pipe 2 and pipe 3 are respectively $V_2 = 1.77\ m\ s^{-1}$ and $V_3 = 1.43\ m\ s^{-1}$. Let the flow in pipe 1 be $Q_1$. The flows in pipe 2 and pipe 2 be $Q_2$ and $Q_3$ respectively.

Applying equation of continuity, we have,

Discharge in pipe 1 = Discharge in pipe 2 + Discharge in pipe 3

i.e., $Q_1 = Q_2 + Q_3$ …… (29)

(a) Diameter, $D_3$, required for $Q_3 = 2Q_2$

Putting $Q_3 = 2Q_2$ in (29), we have,

$$Q_1 = Q_2 + 2Q_2 = 3Q_2 \quad \text{...... (30)}$$

Diameter of pipe 2, $D_2 = 250\ mm = 0.250\ m$

Area of cross-section of pipe 2, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4}(0.25)^2 = 0.0491\ m^2$

Mean velocity of flow in pipe 2, $V_2 = 1.77\ m\ s^{-1}$

Hence, discharge in pipe 2, $Q_2 = A_2 V_2 = (0.04191\ m^2) \times (1.77\ m\ s^{-1})$
$= 0.0869\ m^3\ s^{-1}$

$Q_3 = 2Q_2 = 2 \times 0.0869\ m^3 s^{-1} = 0.174\ m^3 s^{-1}$

$\Rightarrow Q_3 = 0.174\ m^3s^{-1} = A_3V_3 = A_3\,(1.43\ m\ s^{-1})$

$$\Rightarrow A_3 = \frac{Q_3}{V_2} = \frac{0.174\ \mathrm{m^3 s^{-1}}}{1.43\ \mathrm{m\ s^{-1}}} = 0.122\ m^2$$

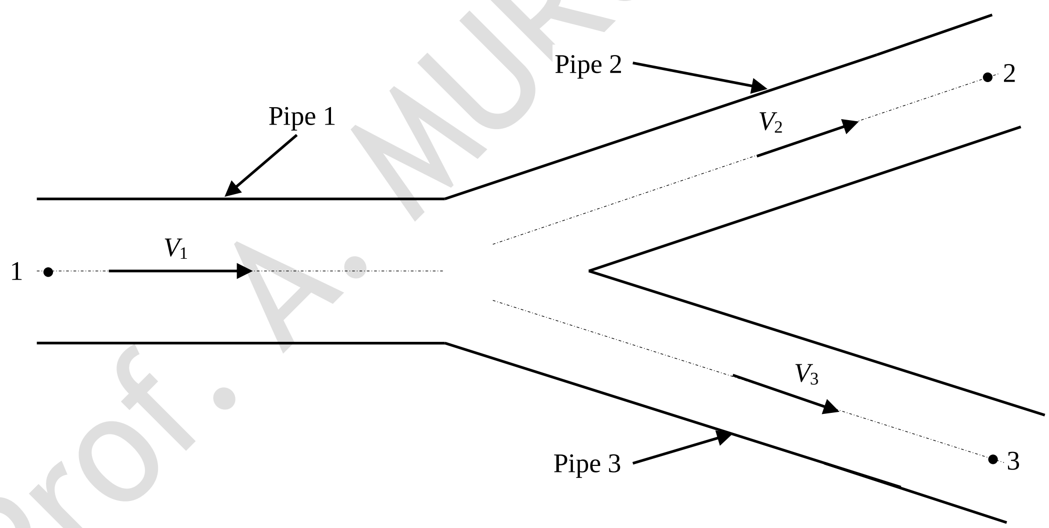
$$\Rightarrow A_3 = 0.122\ m^2 = \frac{\pi}{4} D_3^2$$

$$\Rightarrow D_3 = \sqrt{\frac{4 \times 0.122\ \mathrm{m^2}}{\pi}} = 0.394\ m$$

(b) Total discharge in pipe 1, $Q_1$

$Q_1 = Q_2 + Q_3 = 0.0869\ m^3s^{-1} + 0.174\ m^3s^{-1} = 0.261\ m^3s^{-1}$

**Assignment Problem 1**. Water flows through the branching pipe shown below. Given the following information, find the diameter of the pipe, $D_2$, required at section 2 to maintain continuity of flow.




**Assignment Problem 2**. Two separate pipelines (1 and 2) join together to form a larger pipeline (3). It is known that $D_1 = 0.2$ m, $D_2 = 1.0$ m, $Q_2 = 0.23$ $m^3\ s^{-1}$ and $Q_3 = 0.35$ $m^3\ s^{-1}$ (a) What is the value of $Q_1$, $V_1$ and $V_2$? (b) If $V_3$ must not exceed 3.00 m $s^{-1}$, what is the minimum diameter, $D_3$, which can be used?

**VELOCITY POTENTIAL**

The velocity potential is defined as the scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. Velocity potential is denoted by the Greek symbol ϕ (‘phi’).

Mathematically, the velocity potential for unsteady flow is defined as

$$\phi = f(x, y, z, t)$$

For steady flow, velocity potential is defined as

$$\phi = f(x, y, z)$$

such that

$$-\frac{\partial \phi}{\partial x} = u$$
$$-\frac{\partial \phi}{\partial y} = v$$
$$-\frac{\partial \phi}{\partial z} = w \qquad \ldots\ldots \quad (31)$$

where $u$, $v$ and $w$ are components of velocity in the $x$, $y$ and $z$ directions respectively.

*What is the significance of the negative sign in the above expressions?*

The negative sign denotes that the velocity potential $\phi$ decreases with an increase in the values of $x$, $y$ and $z$. In other words, it indicates that the flow is always in the direction of decreasing $\phi$.

The continuity equation in three dimensions for steady flow of an incompressible fluid is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Putting $u = -\frac{\partial \phi}{\partial x}; v = -\frac{\partial \phi}{\partial y}; w = -\frac{\partial \phi}{\partial z}$ in the above expression, we have,

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

$$\Rightarrow -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \ldots\ldots \quad (32)$$

The above equation is known as the ***Laplace equation***. This may be expressed in vector notation as

$$\nabla^2 \phi = 0$$

It is evident that any function $\phi$ that satisfies the Laplace equation will correspond to some case of fluid flow.

For a rotational flow, the components of rotation are given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Putting $u = -\frac{\partial \phi}{\partial x}; v = -\frac{\partial \phi}{\partial y}; w = -\frac{\partial \phi}{\partial z}$ in the above expressions for $\omega_x$, $\omega_y$ and $\omega_z$ we have,

$$\omega_x = \frac{1}{2}\left\{\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right\} = \frac{1}{2}\left\{-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right\}$$

$$\omega_y = \frac{1}{2}\left\{\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right\} = \frac{1}{2}\left\{-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right\}$$

$$\omega_z = \frac{1}{2}\left\{\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right\} = \frac{1}{2}\left\{-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right\}$$

However, if $\phi$ is a continuous function then

$$\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}; \quad \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}; \quad \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

Therefore, $\omega_x = \omega_y = \omega_z = 0$ that is, the flow is irrotational.

Hence, it can be stated that any function $\phi$ that satisfies the Laplace equation is a possible case of irrotational flow since the continuity equation is satisfied. In other words, velocity potential exists only for irrotational flows. Hence, often an irrotational flow is known as ***potential flow***.

**Example 7.** Determine which of the following fields represent possible example of irrotational flow:

(a) $u = Cx; v = -Cy$

(b) $u = -\frac{Cx}{y}; v = C \log xy$

© $u = \left(Ax^2 - Bxy\right); v = \left(-2Axy + \frac{1}{2}By^2\right)$

**Solution:** If at every point in the flowing fluid the rotation components about the *x*, *y* and *z* axes, $\omega_x$, $\omega_y$, and $\omega_z$ respectively are zero, the flow is said to be irrotational.

The rotation components are given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right); \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right); \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

As $\omega_x = 0$, we have, $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$

As $\omega_y = 0$, we have, $\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \mathbf{0} \Rightarrow \frac{\partial u}{\partial z} = \frac{\partial w}{\partial z}$

As $\omega_z = 0$, we have, $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

(a) $u = Cx; v = -Cy$

$\frac{\partial u}{\partial y} = 0;\ \frac{\partial v}{\partial x} = 0$

Hence, $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$; i.e., $\omega_z = 0$

Hence, $u = Cx; v = -Cy$ represent possible example of irrotational flow.

(b) $u = -\frac{Cx}{y}; v = C\log xy$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{Cx}{y}\right) = \frac{\partial}{\partial y}\left(-Cxy^{-1}\right) = -Cx(-y^{-1-1}) + f(x,z)$$

$$= -Cx\left(-\frac{1}{y^2}\right) + f(x,z)$$

$$= Cx\left(\frac{1}{y^2}\right) + f(x,z)$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(C\log xy) = C\,\frac{\partial}{\partial x}(\log xy) = C\left(\frac{1}{xy}\right)y + f(y,z) = C\left(\frac{1}{x}\right) + f(y,z)$$

As $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$, $\omega_z \neq 0$; hence the velocity fields do not represent possible case of irrotational flow and they represent a possible case of rotational flow.

© $u = \left(Ax^2 - Bxy\right); v = \left(-2Axy + \frac{1}{2}By^2\right)$

$\frac{\partial u}{\partial y} = \left(0 - Bx(1)\right) = -Bx$

$\frac{\partial v}{\partial x} = \left(-2Ay(1) + 0\right) = -2Ay$

As $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$, $\omega_z \neq 0$; hence the velocity fields do not represent possible case of irrotational flow and they represent a possible case of rotational flow.

**Example 8:** Calculate the velocity components *u* and *v* for the following velocity potential functions:

(a) $\phi = x + y$

(b) $\phi = x^2 + y^2$ [**Assignment]**

(c) $\phi = \dfrac{Ax}{x^2 + y^2}$

(d) $\phi = \sin x \sin y$

(e) $\phi = \log(x + y)$ [**Assignment**]

Which of these velocity potential functions satisfy the continuity equation?

**Solution.**

(a) $\phi = \boldsymbol{x} + \boldsymbol{y}$

$$\boldsymbol{u} = -\frac{\partial \phi}{\partial \boldsymbol{x}} = -\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x} + \boldsymbol{y}) = -1$$

$$\boldsymbol{v} = -\frac{\partial \phi}{\partial \boldsymbol{y}} = -\frac{\partial}{\partial \boldsymbol{y}}(\boldsymbol{x} + \boldsymbol{y}) = -1$$

For a steady two – dimensional flow of an incompressible fluid, the Laplace equation must be satisfied. That is, $\dfrac{\partial^2 \phi}{\partial \boldsymbol{x}^2} + \dfrac{\partial^2 \phi}{\partial \boldsymbol{y}^2} = 0$

The continuity equation for steady two-dimensional flow of an incompressible fluid may be written as:

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = 0; \ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}} = 0$$

Hence, $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}} = 0 + 0 = 0$

Therefore, the velocity potential function, $\phi = \boldsymbol{x} + \boldsymbol{y}$, satisfy the continuity equation.

(c) $\phi = \dfrac{\boldsymbol{Ax}}{\boldsymbol{x}^2 + \boldsymbol{y}^2}$

$$\boldsymbol{u} = -\frac{\partial \phi}{\partial \boldsymbol{x}} = -\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\boldsymbol{Ax}}{\boldsymbol{x}^2 + \boldsymbol{y}^2}\right)$$

This is of the form: $-\frac{\partial}{\partial x}\left(\frac{p}{q}\right)$

$$\frac{\partial}{\partial x}\left(\frac{p}{q}\right) = \frac{q\left(\frac{\partial p}{\partial x}\right)-p\left(\frac{\partial q}{\partial x}\right)}{q^2}$$

Here, $p = Ax; q = \left(x^2+y^2\right)$

Hence, $-\frac{\partial}{\partial x}\left(\frac{Ax}{x^2+y^2}\right) = -\frac{\left(x^2+y^2\right)\frac{\partial}{\partial x}(Ax)-Ax\frac{\partial}{\partial x}\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}$

$$= -\frac{\left(x^2+y^2\right)A - Ax(2x)}{\left(x^2+y^2\right)^2}$$

$$= -\frac{Ax^2+Ay^2-2Ax^2}{\left(x^2+y^2\right)^2}$$

$$= \frac{Ax^2-Ay^2}{\left(x^2+y^2\right)^2} = \frac{A\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{Ax}{x^2+y^2}\right) = -\frac{\left(x^2+y^2\right)\frac{\partial}{\partial y}(Ax)-Ax\frac{\partial}{\partial y}\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= -\frac{\left(x^2+y^2\right)0 - Ax(2y)}{\left(x^2+y^2\right)^2}$$

$$= \frac{2Axy}{\left(x^2+y^2\right)^2}$$

***Solution incomplete***

(d) $\phi = \sin x \sin y$

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}(\sin x \sin y)$$

This is of the form: $-\frac{\partial}{\partial x}(pq)$

$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{pq}) = \boldsymbol{p}\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{x}} + \boldsymbol{q}\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}}$$

Here, $\boldsymbol{p} = \sin \boldsymbol{x}; \boldsymbol{q} = \sin \boldsymbol{y}$

Therefore, $-\frac{\partial}{\partial \boldsymbol{x}}(\sin \boldsymbol{x} \sin \boldsymbol{y}) = -\left[\sin \boldsymbol{x}\frac{\partial}{\partial \boldsymbol{x}}(\sin \boldsymbol{y}) + \sin \boldsymbol{y}\frac{\partial}{\partial \boldsymbol{x}}(\sin \boldsymbol{x})\right]$

$$= -[\sin \boldsymbol{x}(0) + \sin \boldsymbol{y}(\cos \boldsymbol{x})]$$

$$= -\cos \boldsymbol{x} \sin \boldsymbol{y}$$

$$\boldsymbol{v} = -\frac{\partial \phi}{\partial \boldsymbol{y}} = -\frac{\partial}{\partial \boldsymbol{y}}(\sin \boldsymbol{x} \sin \boldsymbol{y}) = -\left[\sin \boldsymbol{x}\frac{\partial}{\partial \boldsymbol{y}}(\sin \boldsymbol{y}) + \sin \boldsymbol{y}\frac{\partial}{\partial \boldsymbol{y}}(\sin \boldsymbol{x})\right]$$

$$= -[\sin \boldsymbol{x}(\cos \boldsymbol{y}) + \sin \boldsymbol{y}(0)]$$

$$= -\sin \boldsymbol{x} \cos \boldsymbol{y}$$

***Solution incomplete***

**STREAM FUNCTION**

The stream function is defined as a scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles taken in the counterclockwise direction to this direction. It is denoted by the symbol ψ (Greek 'psi').

Let us consider the stream function for the case of two-dimensional flow. Mathematically, for unsteady flow, stream function may be defined as

$$\psi = f(x, y, t)$$

For steady flow, stream function may be defined as

$$\psi = f(x, y)$$

and

$$\frac{\partial \psi}{\partial x} = v$$

$$\frac{\partial \psi}{\partial y} = -u \qquad \ldots\ldots \quad (33)$$

Let us compare equations (31) and (33)

From (31), $u = -\frac{\partial \phi}{\partial x}$ and $v = -\frac{\partial \phi}{\partial y}$

Putting the expression for u and v from (31) in (33), we have,

$$\frac{\partial \psi}{\partial x} = v = -\frac{\partial \phi}{\partial y} \qquad \ldots.. \quad (34)$$

$$\frac{\partial \psi}{\partial y} = -u = -\left(-\frac{\partial \phi}{\partial x}\right) = \frac{\partial \phi}{\partial x} \qquad \ldots\ldots \quad (34)$$

Equations (34) are known as Cauchy-Rieman equations and they enable the computation of stream function if the velocity potential is known or vice-versa in a potential flow.

Let us substitute the values of u and v from (33) in the expression for rotation component $\omega_z$, we have,

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] \qquad \ldots\ldots \quad (35)$$

Equation (35) is known as Poisson's equation. For an irrotational flow, since $\omega_z = 0$, equation (35) becomes

$$\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] = 0$$

$$\Rightarrow \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] = 0 \qquad \ldots\ldots \quad (36)$$

Equation (36) is known as the Laplace equation for ψ.

Substituting the values of *u* and *v* from equation (33) in the equation of continuity for two-dimensional steady flow of an incompressible fluid, we get,

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$$

This will be true only if $\psi$ is a continuous function and its second derivative exists. Hence, it may be stated that any function ψ which is continuous is a possible case of fluid flow (which may be rotational or irrotational) since the continuity equations is satisfied.

If the stream function ψ satisfies the Laplace equation, then it is a possible case of irrotational flow.

**Example 6.** Which of the following stream function ψ are possible irrotational flow fields?

(a) $\psi = Ax + By^2$

(b) $\psi = Ax^2 y^2$

© $\psi = A\sin xy$

(d) $\psi = A\log\left(\frac{x}{y}\right)$

(e) $\psi = \left(y^2 - x^2\right)$

**Solution.** For a two-dimensional irrotational flow, since the rotation component $\omega_z = 0$, we have,

$$\omega_z = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0$$

$$\Rightarrow \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0$$

The above equation is called the Laplace equation for the stream function ψ of two-dimensional irrotational flow.

(a) $\psi = Ax + By^2$

$$\frac{\partial\psi}{\partial x} = A;\ \frac{\partial^2\psi}{\partial x^2} = 0$$

$$\frac{\partial\psi}{\partial y} = B(2y) = 2By$$

$$\frac{\partial^2\psi}{\partial y^2} = 2B$$

Therefore, $\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0 + 2B \neq 0$

i.e., the given stream function does not represent a possible irrotational flow field.

(b) $\psi = Ax^2 y^2$

$$\frac{\partial \psi}{\partial x} = Ay^2(2x) = 2Axy^2$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2Ay^2(1) = 2Ay^2$$

$$\frac{\partial \psi}{\partial y} = Ax^2(2y) = 2Ax^2 y$$

$$\frac{\partial^2 \psi}{\partial y^2} = 2Ax^2(1) = 2Ax^2$$

Therefore, $\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 2Ay^2 + 2Ax^2 \neq 0$

i.e., the given stream function does not represent a possible irrotational flow field.

© $\psi = A\sin xy$

$$\frac{\partial \psi}{\partial x} = A\cos(xy).y = Ay\cos(xy)$$

$$\frac{\partial^2 \psi}{\partial x^2} = Ay(-\sin xy)y = -Ay^2(\sin xy)$$

$$\frac{\partial \psi}{\partial y} = A\cos(xy)x = Ax\cos(xy)$$

$$\frac{\partial^2 \psi}{\partial y^2} = Ax(-\sin xy)x = -Ax^2 \sin(xy)$$

Therefore, $\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = Ax\cos(xy) + [-Ax^2 \sin(xy)] \neq 0$

i.e., the given stream function does not represent a possible irrotational flow field.

(d) $\psi = A\log\left(\frac{x}{y}\right)$

$$\frac{\partial \psi}{\partial x} = A\left(\frac{1}{x/y}\right)\left(\frac{1}{y}\right) = \frac{A}{y}\left(\frac{y}{x}\right) = \frac{A}{x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{A}{x}\right) = A\frac{\partial}{\partial x}\left(x^{-1}\right) = A(-1)x^{-1-1} = -Ax^{-2} = -\frac{A}{x^2}$$

$$\frac{\partial \psi}{\partial y} = A\left(\frac{1}{x/y}\right)x(-1)(y^{-1-1}) = -A\left(\frac{y}{x}\right)xy^{-2} = -Ax\left(\frac{y}{x}\right)\left(\frac{1}{y^2}\right) = -\frac{A}{y}$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{A}{y}\right) = -A\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -A\frac{\partial}{\partial y}\left(y^{-1}\right) = -A(-1)\,y^{-1-1}$$

$$= Ay^{-2} = \frac{A}{y^2}$$

Therefore, $\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) = -\frac{A}{x^2} + \frac{A}{y^2} \neq 0$

i.e., the given stream function does not represent a possible irrotational flow field.

(e) $\psi = \left(y^2 - x^2\right)$

$$\frac{\partial\psi}{\partial x} = -2\boldsymbol{x}$$

$$\frac{\partial^2\psi}{\partial x^2} = -2$$

$$\frac{\partial\psi}{\partial y} = 2\boldsymbol{y}$$

$$\frac{\partial^2\psi}{\partial y^2} = 2$$

Hence, $\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) = -2 + 2 = 0$

Therefore, the given stream function represents a possible irrotational flow field.

Determine the stream function for each of the following velocity potential functions:

(a) $\phi = x + y$

(b) $\phi = x^2 + y^2$ [**Assignment]**

(c) $\phi = \frac{Ax}{x^2 + y^2}$

(d) $\phi = \sin x \sin y$

(e) $\phi = \log(x + y)$ [**Assignment**]

**Solution.**

(a) $\phi = x + y$

From equation (34),

$$\frac{\partial\psi}{\partial x} = v = -\frac{\partial\phi}{\partial y}$$
$$\frac{\partial\psi}{\partial y} = -u = -\left(-\frac{\partial\phi}{\partial x}\right) = \frac{\partial\phi}{\partial x}$$
$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x+y) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} = 1 + 0 = 1 = \frac{\partial\psi}{\partial y}$$
$$\Rightarrow \partial\psi = \partial y$$
$$\Rightarrow \int\partial\psi = \int\partial y$$
$$\Rightarrow \psi = y + f(x)$$
$$-\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}(x+y) = -\frac{\partial x}{\partial y} - \frac{\partial y}{\partial y} = 0 - 1 = -1 = \frac{\partial\psi}{\partial x}$$
$$\Rightarrow \partial\psi = -\partial x$$
$$\Rightarrow \int\partial\psi = -\int\partial x$$
$$\Rightarrow \psi = -x + f(y)$$

Hence, $\psi = y - x$

(b) $\phi = x^2 + y^2$

$$\frac{\partial\psi}{\partial x} = v = -\frac{\partial\phi}{\partial y}$$
$$\frac{\partial\psi}{\partial y} = -u = -\left(-\frac{\partial\phi}{\partial x}\right) = \frac{\partial\phi}{\partial x}$$
$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2\right) = \frac{\partial\left(x^2\right)}{\partial x} + \frac{\partial\left(y^2\right)}{\partial x} = 2x + 0 = \frac{\partial\psi}{\partial y}$$
$$\Rightarrow \partial\psi = 2x.\partial y$$
$$\Rightarrow \int\partial\psi = \psi = 2\int x\partial y = 2x\int\partial y = 2xy + f(x)$$
$$-\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}\left(x^2 + y^2\right) = -\frac{\partial\left(x^2\right)}{\partial y} - \frac{\partial\left(y^2\right)}{\partial y} = 0 - 2y = \frac{\partial\psi}{\partial x}$$
$$\Rightarrow \partial\psi = -2y.\partial x$$
$$\Rightarrow \int\partial\psi = \psi = -2y\int\partial x = -2yx + f(y)$$
$$\Rightarrow \psi = 2xy - 2xy + f(x) + f(y) = 0 + f(x) + f(y)$$

The velocity potential function defined by (b) does not represent the flow field and hence there is no explicit stream function corresponding to this velocity potential.

(c) $\phi = \dfrac{Ax}{x^2 + y^2}$

$$\frac{\partial \psi}{\partial x} = v = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \psi}{\partial y} = -u = -\left(-\frac{\partial \phi}{\partial x}\right) = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{Ax}{x^2 + y^2}\right) = \frac{\left(x^2 + y^2\right)A - (Ax)(2x)}{\left(x^2 + y^2\right)^2} = \frac{A\left(x^2 + y^2 - 2x^2\right)}{\left(x^2 + y^2\right)^2} = \frac{A\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} = \frac{\partial \psi}{\partial y}$$

$$\Rightarrow \partial \psi = \frac{A\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}\partial y$$

$$\Rightarrow \int \partial \psi = \int \frac{A\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}\partial y$$

$$\Rightarrow \psi = A\int \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\partial y$$

***Solution incomplete***