HYDROSTATICS

TOTAL PRESSURE

When a static mass of fluid comes in contact with a plane or curved surface, the fluid exerts a force on the surface. This force is called ***total pressure***.

What is the direction in which this total pressure exerted by the fluid acts on the surface?

When a fluid is at rest, no tangential force exists on the fluid. That is, when a fluid is at rest, it cannot sustain any shear forces. Hence, the fluid at rest exerts a force in a direction normal to the surface with which it comes in contact.

CENTRE OF PRESSURE

The point of application of total pressure on the surface is called *centre of pressure.*

TOTAL PRESSURE ON A PLANE SURFACE FULLY IMMERSED IN LIQUID AND HELD HORIZONTAL

Let us consider a plane surface of area A immersed fully in a static mass of liquid of specific weight γ. The plane surface is held such that it is in a horizontal position (parallel to the free liquid surface) at a depth h below the free surface of liquid a shown in Figure 1.

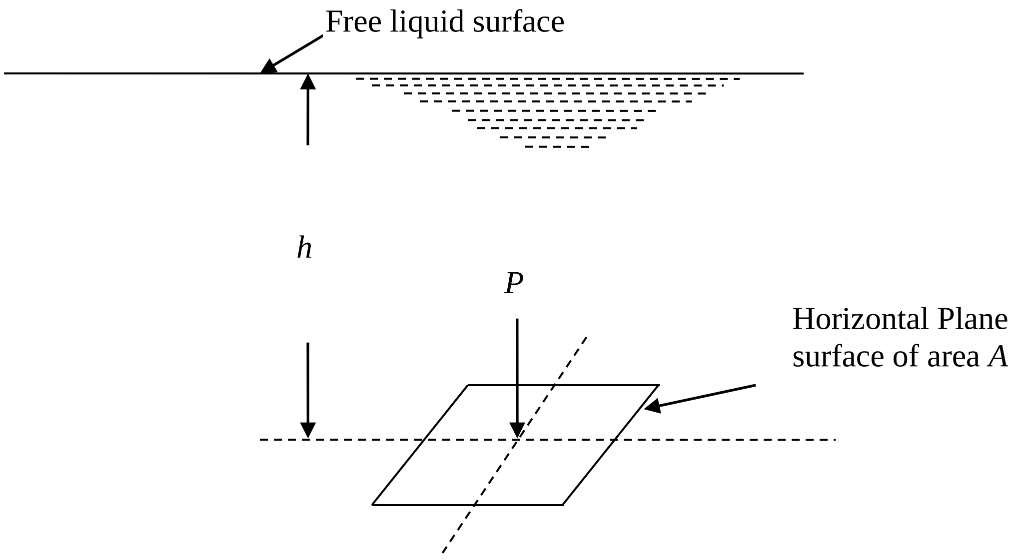


Figure 1 Total Pressure on a Horizontal Plane Surface

As every point on the horizontal surface is at the same depth h below the free surface of liquid, the pressure intensity p over the entire surface is constant equal to γh. Therefore, the total pressure force P acting on the surface is given by

P = pressure intensity over the entire surface x area of the surface
= p x A

$$P = (\gamma h)A = \gamma Ah \qquad \ldots\ldots \; (1)$$

The total pressure force acts in a direction normal (perpendicular) to the horizontal plane surface. It acts in the vertical downward direction through the centroid of the surface.

TOTAL PRESSURE AND CENTRE OF PRESSURE FOR A PLANE SURFACE FULLY IMMERSED IN LIQUID AND HELD VERTICAL

Figure below shows a plane surface of arbitrary shape and area A, fully immersed in a static mass of liquid of specific weight γ. The plane surface is held vertical such that the centroid of the surface is located at a vertical depth $\bar{x}$

below the free surface of liquid. Let us determine the magnitude and location of point of application of the total pressure force acting on the vertical plane surface.

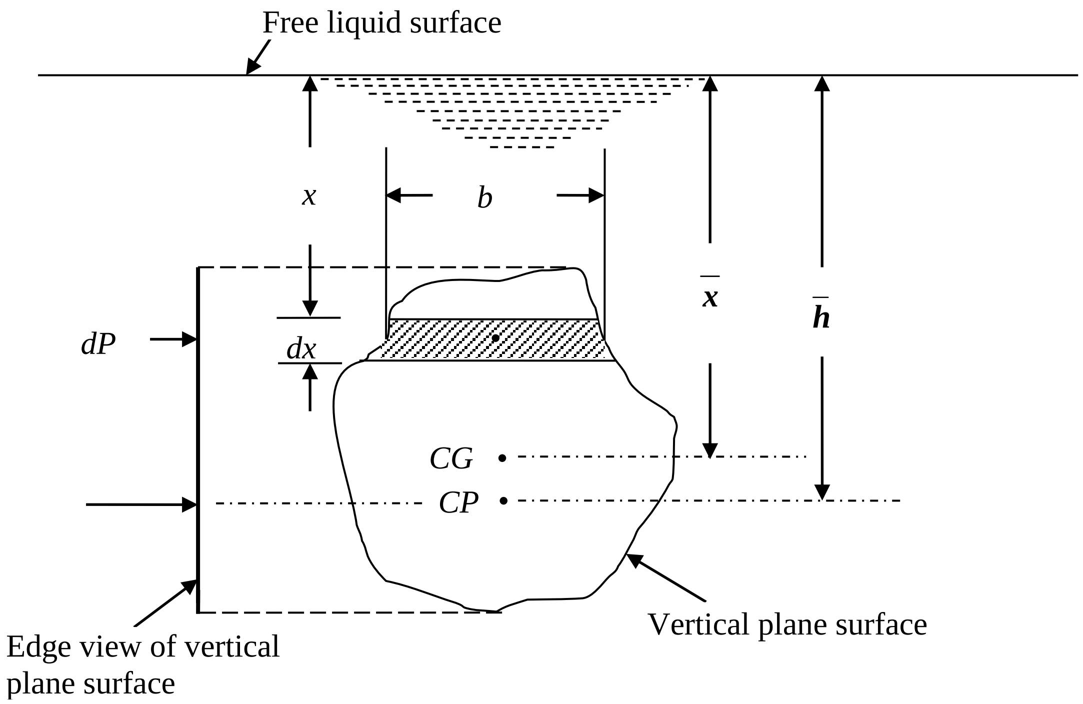


Figure 2 Total Pressure on a Vertical Plane Surface Immersed Fully in a Liquid

Here, the pressure intensity over the entire surface is not constant, that is, it varies from point to point on the surface. Why?

As the depth of liquid varies from point to point on the surface, the pressure intensity is not constant over the entire surface.

Let us now determine the magnitude of total pressure force P acting on the vertical plane surface. For this purpose, let us divide the entire surface into a number of elementary strips that are parallel to one another. Let us compute the magnitude of total pressure on each of these elementary strips. The summation of these total pressures on these small strips gives the magnitude of total pressure on the complete surface.

Let us consider an elementary strip of width b and thickness dx located at a vertical depth x below the free surface of liquid. As the thickness of the elementary strip considered is too small, let us assume that the pressure intensity over this strip is uniform and does not vary. Let the uniform pressure intensity over the entire strip be p whose magnitude is equal to specific weight times the depth of the strip below the free surface, that is $p = \gamma x$. Let the area of the strip be represented as dA.

dA = (width of strip) x (thickness of strip) = b x dx

Total pressure on the strip, dP = (pressure intensity over the strip) x (area of the strip)

$= p \times dA$

$= (\gamma x) \times (b \times dx)$

Therefore, total pressure on the entire surface can be obtained by integrating the expression for dP. That is,

$$P = \int dP = \int (\gamma x)(b.dx)$$
$$= \gamma \int x(b.dx)$$

What the quantity $x(b.dx)$ *represents?*

$(b.dx)$ is the area of the elementary strip of width b and thickness dx.
$x(b.dx)$ represents the first moment of the area of the elementary strip about the axis *OO*. It should be noted that axis *OO* is obtained by the intersection of the free surface of liquid with the vertical plane in which the plane surface lies.
$\int x(b.dx)$ represents the sum of the first moments of areas of elementary strips covering the entire vertical plane surface about the axis *OO*. From the principle of mechanics, this is equal to the product of the area A of the entire vertical plane surface and the vertical distance $\bar{x}$ of the centroid of the vertical plane surface below the free surface of liquid or axis *OO*. That is,

$$\int x(b.dx) = A\bar{x}$$

Hence, $P = \gamma \int x(b.dx) = \gamma A \bar{x}$ …… (2)

Equation (2) represents a general expression for the total pressure exerted by a liquid on a plane surface.

In expression (2), as the quantity $(\gamma \bar{x})$ represents the magnitude of pressure intensity at the centroid of the plane surface, it can be stated that the total pressure on a plane surface is equal to the product of the area of the surface and the intensity of pressure at the location of the centroid of the area.

Can you determine the magnitude of total pressure on a horizontal surface through equation (2)?

Yes. In a horizontal surface, the centroid of the surface $\bar{x}$ is located at a vertical distance h below the free surface of liquid. That is, $\bar{x} = h$.

Centre of Pressure

The point of application of centre of pressure on a plane surface is known as centre of pressure.

In case of a horizontal plane surface immersed in a liquid, the centre of pressure coincides with the centroid of the surface. *Why*?

As the pressure intensity throughout the surface is uniform (constant), the total pressure would pass through the centroid of the surface.

In case of a vertical plane surface immersed fully in a liquid, the centre of pressure does not coincide with the centroid of the surface. *Why*?

As the pressure intensity increases with increase in the depth of liquid, the centre of pressure lies below the centroid of the area.

Let us now determine the location of the centre of pressure for a vertical plane surface immersed in a liquid. Let $\bar{\boldsymbol{h}}$ be the vertical depth of the location of the centre of pressure below the free surface of liquid. The moment caused by the total pressure force P about the axis OO is $\mathrm{P}\bar{\boldsymbol{h}}$.

As evaluated earlier, the total pressure on the elementary strip of width b and thickness dx is give by

$dP = (\gamma x) \text{ x } (b \text{ x } dx)$

The moment caused by dP about the axis OO is given by

$(dP)\, x = [(\gamma x) \text{ x } (b \text{ x } dx)] \text{ x } x = \gamma x^2\, (b.dx)$

Likewise, by considering all elementary strips of the entire vertical plane surface and summing up the moments caused by the total pressures on these strips about the axis OO, we have,

$$\int (dP)x = \int \gamma x^2 (b.dx) = \gamma \int x^2 (b.dx)$$

The quantity $x^2(b.dx)$ represents the second moment of the area dA (= $b.dx$) of the elementary strip about the axis OO. The quantity $\int x^2(b.dx)$ represents the sum of the second moments of areas of elementary strips covering the entire surface about the axis OO. This is equal to the moment of inertia I_0 of the plane surface about the axis OO. That is,

$$I_0 = \int x^2(b.dx) \qquad \ldots\ldots \quad (3)$$

The quantity $\gamma\int x^2(b.dx)$ represents the sum of moments of total pressures on elementary strips covering the entire surface about the axis OO. By the principle of moments, this is equal to the moment caused by the total pressure P acting on the entire plane surface about the axis OO. The moment caused by P about the axis OO is given by $P\bar{h}$.

We have, $P\bar{h} = \gamma\int x^2(b.dx) \qquad \ldots\ldots \quad (4)$

Introducing equation (3) in (4), we have,

$$P\bar{h} = \gamma I_0$$

$$\Rightarrow \bar{h} = \frac{\gamma I_0}{P}$$

Substituting $P = \gamma A\bar{x}$, from equation (2) in above expression, we have,

$$\bar{h} = \frac{\gamma I_0}{\gamma A\bar{x}} = \frac{I_0}{A\bar{x}} \qquad \ldots\ldots \quad (5)$$

From the "Parallel Axes Theorem" for moment of inertia, we have,

$$I_0 = I_G + A\bar{x}^2 \qquad \ldots\ldots \quad (6)$$

where I_G = moment of inertia of the plane surface about an axis passing through the centroid of the area and parallel to the axis OO.

Introducing equation (6) in (5), we have,

$$\bar{h} = \frac{I_G + A\bar{x}^2}{A\bar{x}} = \frac{I_G}{A\bar{x}} + \frac{A\bar{x}^2}{A\bar{x}} = \frac{I_G}{A\bar{x}} + \bar{x}$$

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}} \qquad \ldots\ldots \quad (7)$$

Equation (7) gives the position of centre of pressure for a plane surface immersed vertically in a static mass of liquid.

Let us Mathematically prove that *the centre of pressure for a vertical plane surface is always below the centroid of the surface.*

In equation (7), the quantity $\frac{I_G}{A\bar{x}}$ is always positive. Hence, the value of $\bar{h}$ is $\bar{x}$ plus a positive quantity. This shows that $\bar{h}$ is always greater than $\bar{x}$. That is, the position of centre of centre of pressure is always below the centroid of the vertical plane surface.

When the vertical plane surface is immersed deeper and deeper below the liquid surface, the centre of pressure becomes closer and closer to the centroid of the surface. *Why?*

As the vertical surface is immersed deeper and deeper, the vertical depth $\bar{x}$ of the centroid of the surface becomes greater and greater. Hence, the quantity $\frac{I_G}{A\bar{x}}$ becomes lesser and lesser. As $\frac{I_G}{A\bar{x}}$ becomes lesser and lesser, $\bar{h}$ becomes closer and closer to $\bar{x}$.

Example 1. Determine the total force and location of center of pressure for the cases shown in Figure.

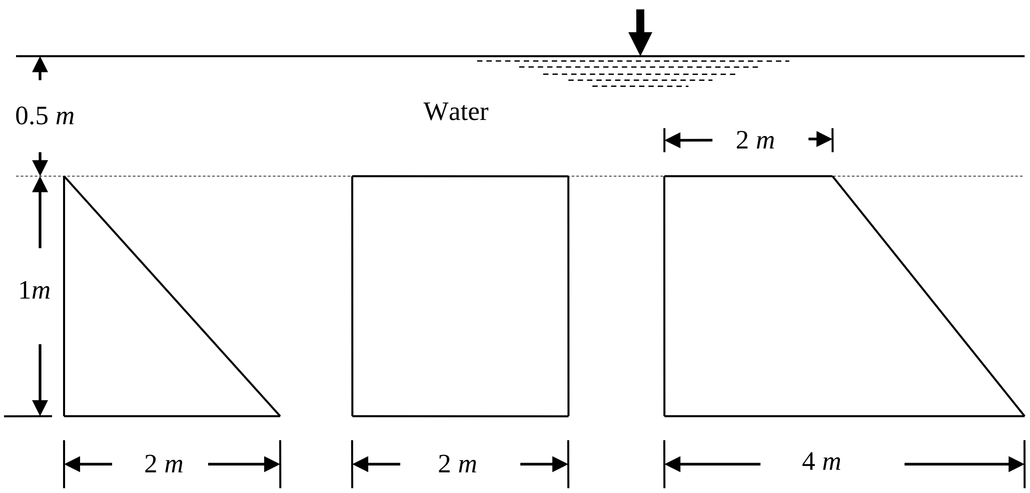


Figure Example 1

Solution.

Case (a) Right – angled triangular plate immersed vertically in water with its vertex at a depth 0.5 m below the free surface of water.

Required: Force exerted by water on the plate, P = ?
Position of center of pressure (CP) = ?

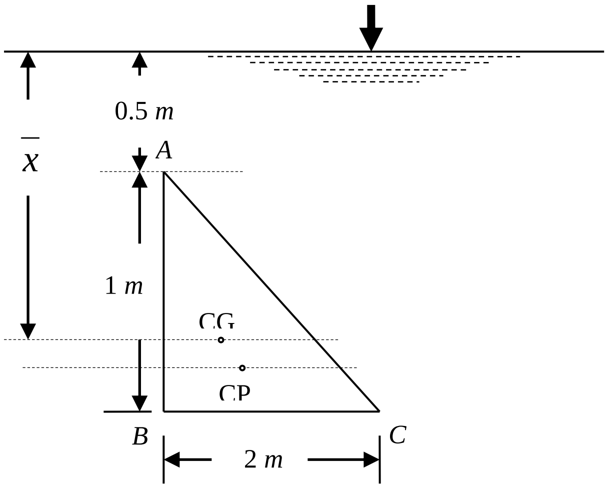


Figure Example 1 - Case (a)

Force exerted by water on the plate, $P = \gamma A\bar{x}$

where, γ = specific weight of water = 9810 N/m^3

A = area of triangular plate in contact with water

$= \frac{1}{2} \text{x(base)x(height)}$

$= \frac{1}{2} \text{ x } \overline{BC} \text{ x } \overline{AB} = \frac{1}{2} \text{ x } 2\,\text{m x } 1\,\text{m} = 1\ m^2$

$\bar{x}$ = vertical location of centroid of the triangular plate below the free surface of water

$= 0.5\ m + \left(\frac{2}{3}\right)(1\ m) = 1.167\ m$

Hence, $P = (9810\ N/m^3) \text{ x } (1\ m^2) \text{ x } (1.167\ m) = 11445\ N =$ **11.445 *kN***

Position of center of pressure (CP):

Let $\bar{h}$ be the vertical location of the center of pressure (CP) on the triangular plate below the free surface of water.

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

where, I_G = moment of inertia of the triangular plate about an axis passing through the centroid of the plate parallel to the free surface of water

$$= \frac{(2m)(1m)^3}{36} = 0.056\ m^4$$

Hence, $\bar{\boldsymbol{h}} = (1.167m) + \dfrac{0.056m^4}{(1m^2)(1.167m)}$ = 1.167 m + 0.048 m = **1.215 m**

Case (b) Rectangular plate immersed vertically in water with its top edge at a depth 0.5 m below the free surface of water.

Required: Force exerted by water on the plate, P = ?
Position of center of pressure (CP) = ?

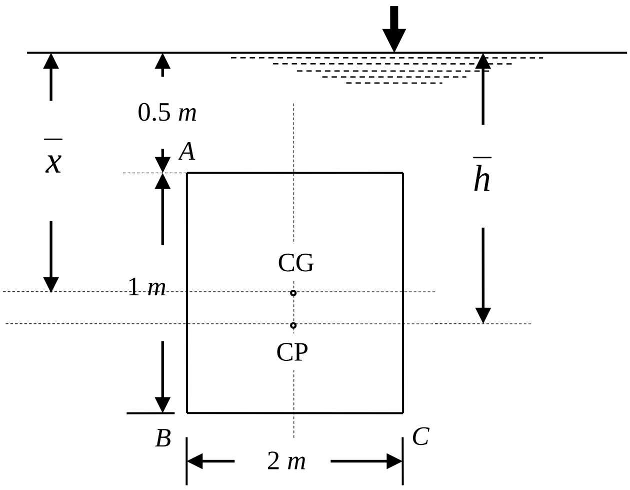


Figure Example 1 - Case (b)

Force exerted by water on the plate, $P = \boldsymbol{\gamma A \bar{x}}$

where, γ = specific weight of water = 9810 N/m^3
A = area of rectangular plate in contact with water
= 2 m x 1 m = 2 m^2
$\bar{\boldsymbol{x}}$ = vertical location of centroid of the rectangular plate below the free surface of water

$$= 0.5 + \left(\frac{1}{2}\right) = 1\ m$$

Hence, P = (9810 N/m^3) x (2 m^2) x (1 m) = 19620 N = **19.62 *kN***

Position of center of pressure (CP):

Let $\bar{h}$ be the vertical location of the center of pressure (CP) on the rectangular plate below the free surface of water.

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

where, I_G = moment of inertia of the rectangular plate about an axis passing through the centroid of the plate parallel to the free surface of water

$$= \frac{(2m)(1m)^3}{12} = 0.167\ m^4$$

Hence, $\bar{h} = 1\,m + \dfrac{0.167\ m^4}{(2\,m^2)(1\,m)} = 1\ m + 0.083\ m = \mathbf{1.083\ m}$

Case (c) Trapezoidal plate immersed vertically in water with its top edge at a depth 0.5 m below the free surface of water.

Required: Force exerted by water on the plate, P = ?
Position of center of pressure (CP) = ?

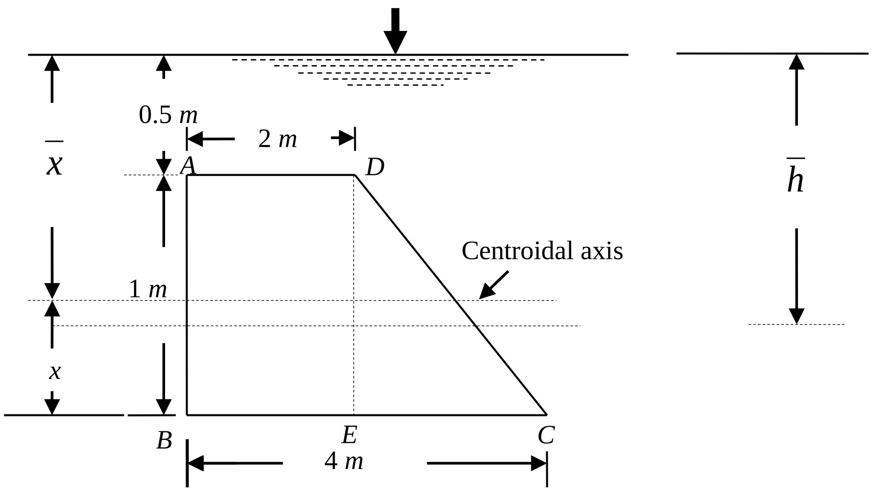


Figure Example 1 - Case (c)

Force exerted by water on the plate, $P = \gamma A\bar{x}$

where, γ = specific weight of water = 9810 N/m^3

A = area of trapezoidal plate in contact with water

$$= \left(\frac{\overline{AD} + \overline{BC}}{2}\right)\overline{AB} = \left(\frac{2\,m + 4\,m}{2}\right)(1\,m) = 3\ \text{m}^2$$

$\bar{x}$ = vertical location of centroid of the trapezoidal plate below the free surface of water

To determine $\bar{x}$, let us divide the trapezium into a rectangle ADEB and a triangle DCE. Let x_1 and x_2 be the vertical locations of centroids of rectangle ADEB and triangle DCE respectively. Let a_1 and a_2 be the areas of rectangle ADEB and triangle DCE respectively.

Using Varignon's theorem, we have,

$$A\bar{x} = a_1 x_1 + a_2 x_2$$

a_1 = 2 m x 1 m = 2 m^2

$a_2 = \frac{1}{2}$ x 2 m x 1 m = 1 m^2

x_1 = vertical location of centroid of the rectangular portion ADEB below the free surface of water

$= 0.5 + \left(\frac{1}{2}\right) = 1\ m$

x_2 = vertical location of centroid of the triangular portion DCE below the free surface of water

$= 0.5\ m + \left(\frac{2}{3}\right)(1\ m) = 1.167\ m$

Hence, we have,

$$\left(3\,m^2\right)\bar{x} = \left(2\,m^2\right)(1\,m) + \left(1\,m^2\right)(1.167\,m)$$

$$\Rightarrow\ \bar{x} = \frac{\left(2\,m^2\right)(1\,m) + \left(1\,m^2\right)(1.167\,m)}{3\,m^2} = 1.056\ \text{m}$$

Hence, $P = (9810\ N/m^3) \times (3\ m^2) \times (1.056\ m) = 31078\ N =$ **31.08 *kN***

Position of center of pressure (CP):

Let $\bar{h}$ be the vertical location of the center of pressure (*CP*) on the trapezoidal plate below the free surface of water.

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

where, I_G = moment of inertia of the trapezoidal plate about an axis passing through the centroid of the plate parallel to the free surface of water

$$I_G = I_{XX1} + I_{XX2}$$

where I_{XX1} = moment of inertia of the rectangular portion *ADEB* about an axis, parallel to the free surface of water, passing through the centroid of the trapezoidal plate.

I_{XX2} = moment of inertia of the triangular portion *DCE* about an axis, parallel to the free surface of water, passing through the centroid of the trapezoidal plate.

Let *x* be the height of the centroidal axis, of the trapezoidal plate, parallel to the free surface of water = (0.5 *m* + 1 *m*) - $\bar{x}$ = 1.5 *m* – 1.056 *m* = 0.444 *m*

$$I_{XX1} = \frac{2 \text{ x } 1^3}{12} + (2 \text{ x } 1)(0.5 - 0.444)^2 = 0.173 \text{ m}^4$$

$$I_{XX2} = \frac{2 \text{ x } 1^3}{36} + \left(\frac{1}{2} \text{ x } 2 \text{ x } 1\right)\left(0.444 - \frac{1}{3}\right) = 0.068 \text{ m}^4$$

$$I_G = 0.173 \text{ m}^4 + 0.068 \text{ m}^4 = 0.241 \text{ m}^4$$

Hence, $\bar{h} = 1.056\, m + \dfrac{0.241\, m^4}{(3\, m^2)(1.056\, m)}$ = 1.056 *m* + 0.076 *m* = **1.132 *m***

Assignment Problem 1. A triangular plate which has a base of 1.5 *m* and an altitude of 2 *m* lies in a vertical plane. The vertex of the gate is 1 *m* below the surface in a tank which contains oil of specific gravity 0.8. Find the force exerted by the oil on the gate and the position of center of pressure.

Example 2. A vertical gate 5 *m* high and 3 *m* wide closes a tunnel running full with water. The pressure at the bottom of the gate is 195 $kN\, m^{-2}$. Determine the total pressure on the gate and the location of the centre of pressure.

Solution.

As the tunnel runs full with water, it acts as a pressure pipe. Water is carried under pressure through the tunnel. The pressure at the bottom of the rectangular gate is 195 $kN\, m^{-2}$.

If the rectangular gate is considered to be immersed vertically in a static mass of water, to exert a pressure equal to 195 $kN\, m^{-2}$ at the bottom of gate, the vertical depth of the bottom of gate below the free surface of liquid is given by

$$h = \frac{p_{bottom}}{\gamma_w} = \frac{195 \text{ x } 10^3\ N\, m^{-2}}{9810\ N\, m^{-3}} = 19.878\ m$$

This is represented in Figure ( ).

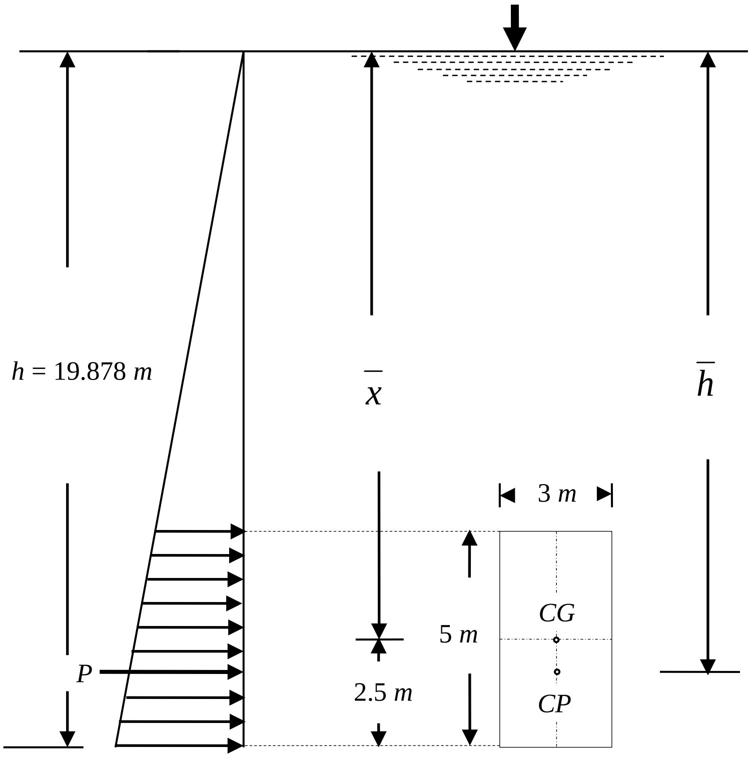


Figure Example 2

Area of gate, $A = 3\ m \times 5\ m = 15\ m^2$

$\bar{x}$ = vertical location of centre of gravity of the rectangular gate below the free surface of water = $h - \dfrac{\text{height of gate}}{2} = 19.878\ m - \dfrac{5\,m}{2}$

$= 19.878$ m – 2.5 m = 17.378 m

Total pressure on the gate, $P = \gamma A \bar{x}$

$= 9810\ N\ m^{-3} \times 15\ m^2 \times 17.378\ m$

$= 2557173\ N = 2.557\ MN$

Location of centre of pressure:

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

I_G = moment of inertia of the rectangular gate about the centroidal axis parallel to the width of the gate

$$= \frac{\text{width x (height)}^3}{12} = \frac{3\,m \times (5\,m)^3}{12} = 31.25\ m^4$$

Hence, $\bar{h} = 17.378\ m + \dfrac{31.25\ m^4}{(15\ m^2)\ \text{x}\ (17.378\ m)}$ = 17.498 *m*

Example 3. A cylindrical vessel of 1 *m* diameter has water in it to a depth of 2 *m*. Oil of specific gravity 0.75 is kept over water column for another 1 *m*. Above the oil, a dead weight of 5000 *N*, base diameter 1 *m* is kept. Estimate the total force and the centre of pressure on a gate 30 *cm* in diameter placed along the vertical surface of the vessel. The lowest point of the gate is on the base of the vessel.

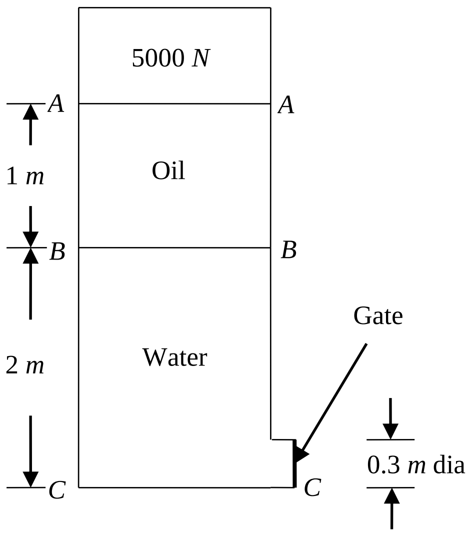


Figure Example 2

Uniform pressure on the surface of oil due to the dead weight of 5000 N having a base of diameter 1 m is given by

$$p_A = \frac{5000\ N}{\text{Area of base of dead weight}} = \frac{5000\ N}{\frac{\pi}{4}(1\,m)^2} = 6363.6\ N/m^2$$

Pressure at the interface of overlying oil and underlying water,

$$\begin{aligned} p_B = p_A + \gamma_{oil} h_{AB} &= 6363.6 + (S_{oil}\gamma_w) h_{AB} \\ &= 6363.6 + (0.75 \text{ x } 9810)(1) \\ &= 6363.6 + 7357.5 \\ &= 13721.1\ N/m^2 \end{aligned}$$

Pressure at the base of the cylindrical vessel,

$$p_c = p_B + \gamma_w h_{BC} = 13721.1 + (9810 \text{ x } 2) = 13721.1 + 19620 = 33341.1\ N/m^2$$

If it is considered that this pressure of 33341.1 N/m^2 is exerted by water column alone, then we should have a column of water of height given by

$$h = \frac{p_C}{\gamma_w} = \frac{33341.1}{9810} = 3.40\ m \text{ above the base of the vessel}$$

This can be represented as shown in Figure below

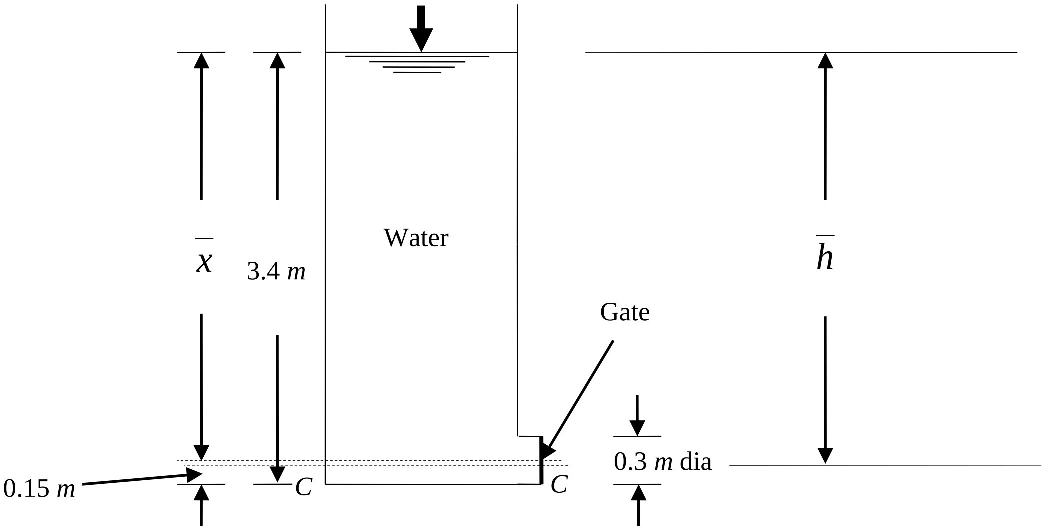


Figure Example 3 - Solution

Total pressure on the ciruclar gate of diameter 0.3 m is given by

$$P = \gamma A \bar{x}$$

where, γ = specific weight of water = 9810 N/m^3
A = area of circular gate in contact with water
$= \frac{\pi}{4}(0.3\,m)^2 = 0.0707\ m^2$
$\bar{x}$ = vertical location of centroid of the rectangular plate below the free surface of water
$= 3.4 - \frac{0.3}{2} = 3.4 - 0.15 = 3.25\ m$

Hence, $P = (9810\ N/m^3) \times (0.0707\ m^2) \times (3.25\ m) = 2254\ N =$ **2.254 kN**

Position of centre of pressure (CP):

Let $\bar{h}$ be the vertical location of the centre of pressure (CP) on the rectangular plate below the free surface of water.

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

where, I_G = moment of inertia of the circular gate about an axis passing through the centroid of the gate parallel to the free surface of water

$$= \frac{\pi}{64}(0.3)^4 = 0.00442\ m^4$$

Hence, $\bar{h} = 3.25\,m + \dfrac{0.00442\ m^4}{(0.0707\ m^2)(3.25\,m)} = 3.25\ m + 0.0192\ m = \mathbf{3.269}\ \boldsymbol{m}$

Assignment Problem 2. A circular gate in a vertical wall has a diameter of 4 *m*. The water surface on the upstream side is 8 *m* above the top of the gate and on the downstream side 1 *m* above the top of the gate. Find the forces acting on the two sides of the gate and the resultant force acting on the gate and its location.

Example 4. A rectangular gate covering an opening 3 *m* wide and 2 *m* high in a vertical wall is hinged about its vertical edge by two pivots placed symmetrically 0.25 *m* from either end. The door is locked by a clamp placed at the centre of the vertical edge. Determine the reactions at the two hinges and the clamp, when the height of water is 1.5 *m* above the top edge of the opening.

Solution.

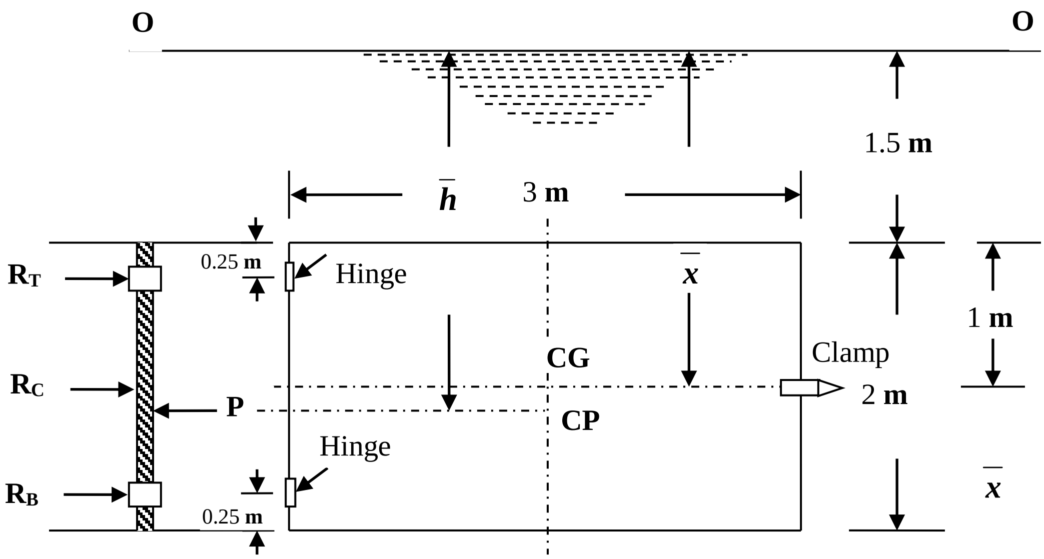


Figure Example 4

Total pressure on the door, $P = \gamma A\bar{x}$

where, γ = specific weight of water = 9810 N/m^3

A = area of door = 3 m x 2 m = 6 m^2

$\bar{x}$ = vertical depth of centroid CG of door below the free surface of water

$= 1.5\ m + \left(\frac{2\ \mathbf{m}}{\mathbf{2}}\right) = 1.5\ m + 1\ m = 2.5\ m$

Therefore, $P = (9810\ N/m^3)\ (6\ m^2)\ (2.5\ m) = 147150\ N$

Location of centre of pressure, $\bar{h}$

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

where I_G = moment of inertia of rectangular door about an axis passing through the centroid parallel to the axis OO

$= [\text{width x (depth)}^3] / 12 = [3\ m \times (2\ m)^3] / 12 = 2\ m^4$

$\bar{h} = 2.5\ \text{m} + \{2\ \text{m}^4 / (6\ \text{m}^2 \times 2.5\ \text{m})\} = 2.633\ \text{m}$

This total pressure P has to be resisted by the reactions offered by the top hinge, bottom hinge and the clamp. Let the reaction offered by the top hinge be denoted as R_T. Let the reaction offered by the bottom hinge be denoted as R_B. Let the reaction offered by the clamp be denoted as R_C.

We have, $P = R_T + R_B + R_C$

By symmetry, half the total pressure P will be resisted by the reaction offered by the clamp. The other half of the total pressure P will jointly be resisted by the reactions offered by the two hinges (top and bottom hinges). That is,

$R_C = P/2 = 147150 / 2 = 73575\ N$

$(R_T + R_B) = P/2 = 147150 / 2 = 73575\ N$

To solve for the magnitudes of reactions R_T and R_B, let us take moments of forces about the top hinge and equate the algebraic sum of moments to zero. While taking moments of forces, the sign convention adopted is that clockwise moments are treated as positive quantities and anti-clockwise moments are treated as negative quantities.

$$- R_C \times (1\ m - 0.25\ m) - R_B \times (2\ m - 0.25\ m - 0.25\ m) + P\ (\bar{h} - 1.5\ m - 0.25\ m) = 0$$

$\Rightarrow - (73575\ N \times 0.75\ m) - (R_B \times 1.5\ m) +$

$147150\ N \times (2.633\ m - 1.5\ m - 0.25\ m) = 0$

$\Rightarrow - 55181.25\ N\,m - 1.5\ R_B + 129933.45\ N\,m = 0$

$\Rightarrow 1.5\ R_B = - 55181.25\ N\,m + 129933.45\ N\,m = 74752.2\ N\,m$

$\Rightarrow R_B = 74752.2\ N\,m / 1.5 = 49834.8\ N$

Hence, $R_T = 73575\ N - R_B = 73575\ N - 49834.8\ N = 23740.2\ N$

Example 5. A vertical rectangular gate 4 *m* x 2 *m* is hinged at a point 0.25 *m* below the centre of gravity of the gate. If the total depth of water is 7 *m*, what horizontal force must be applied at the bottom of gate to keep it closed?

Solution.

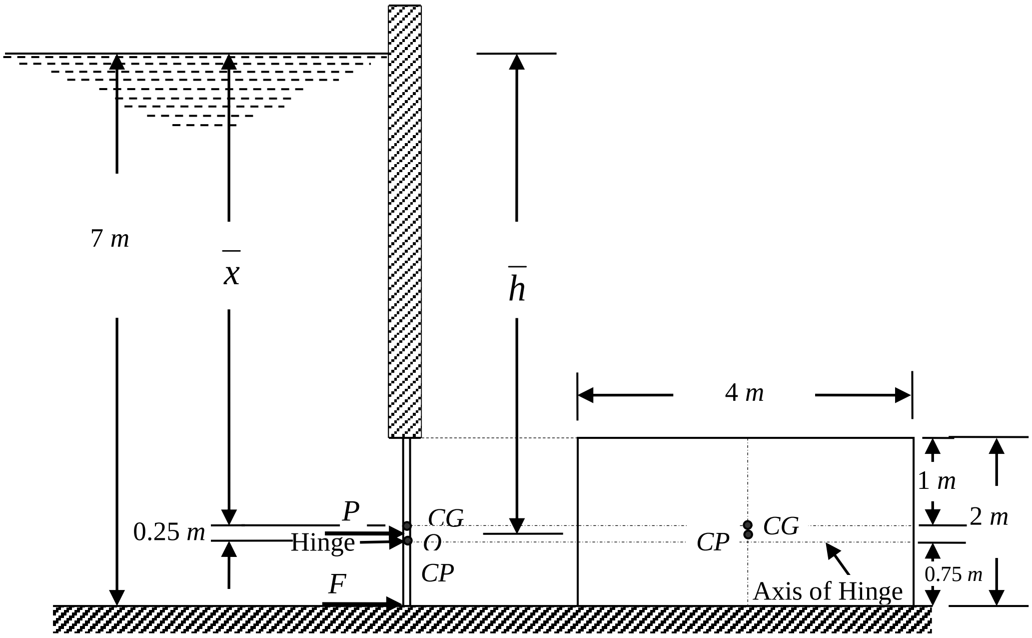


Figure Example 5

Total pressure on the door, $P = \gamma A \bar{x}$

where, γ = specific weight of water = 9810 N/m^3

A = area of rectangular gate = 4 *m* x 2 *m* = 8 m^2

$\bar{x}$ = vertical depth of centroid CG of rectangular gate below the free surface of water

$= 7\ m - \left(\frac{2\ \mathbf{m}}{\mathbf{2}}\right) = 7\ m + 1\ m = 6\ m$

Therefore, $P = (9810\ N/m^3)\ (8\ m^2)\ (6\ m) = 470880\ N = 47.089\ kN$

Position of centre of pressure (CP):

Let $\bar{h}$ be the vertical location of the centre of pressure (*CP*) on the rectangular gate below the free surface of water.

$$\bar{h} = \bar{x} + \frac{I_G}{A\bar{x}}$$

where, I_G = moment of inertia of the rectangular gate about an axis passing through the centroid of the gate parallel to the free surface of water

$$= \frac{(4\,m)\,\text{x}\,(2\,m)^3}{12} = 2.667\ m^4$$

Hence, $\bar{h} = 6\,m + \dfrac{2.667\,m^4}{(8\,m^2)(6\,m)} = 6\ m + 0.056\ m = 6.056\ m$

The hinge is located at a vertical distance 0.25 *m* below the *CG* of the gate; or in other words, it is located at a vertical distance of ($\bar{x}$ + 0.25 m) below the free surface, that is, (6 *m* + 0.25 *m*) = 6.25 *m* below the free surface. As the location of *CP* is 6.056 *m* below the free surface, the hinge is located at (6.25 *m* – 6.056 *m*) = 0.194 *m* below the *CP*. Since the line of action of P is above the hinge '*O*', the gate will open such that it causes a clock-wise moment about '*O*'. Hence, in order to keep the gate closed, the horizontal force F that must be applied at the bottom of the gate must be such that it should cause a counter clock-wise moment about the hinge '*O'*. This necessitates that the line of action of *F* is in the same direction and parallel to *P*.

For static equilibrium of the gate in closed position, the algebraic sum of moments caused by the forces about the hinge must be equal to zero. Taking moments of the forces about the hinge '*O*' and equating the algebraic sum of moments to zero,

$- F \text{ x } (0.75\ m) + P \text{ x } (0.194\ m) = 0$

$\Rightarrow - F \text{ x } (0.75\ m) + (47.089\ kN) \text{ x } (0.194\ m) = 0$

$\Rightarrow F = \dfrac{9.1353\ kN\ m}{0.75\ m} = 12.18\ kN$

TOTAL PRESSURE AND CENTRE OF PRESSURE FOR AN INCLINED PLANE SURFACE

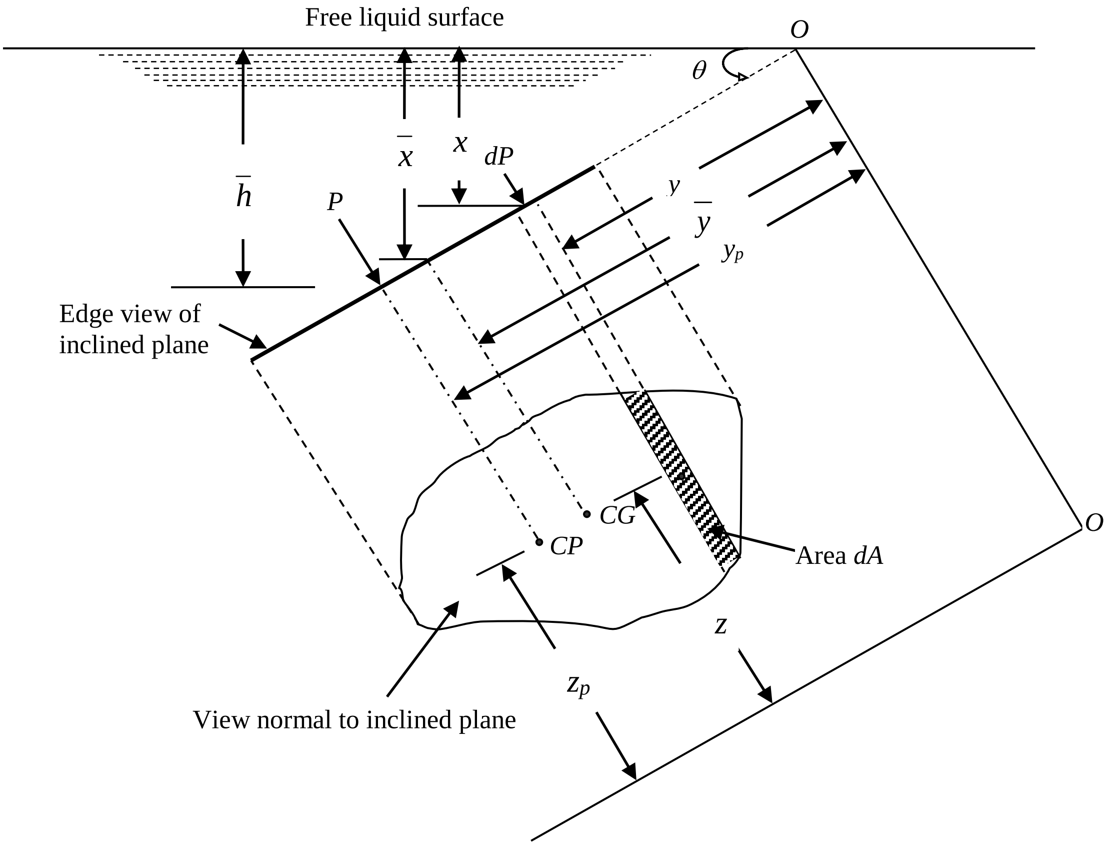


Figure Total Pressure and Centre of Pressure on an Inclined Surface

Let us consider a plane surface of arbitrary shape and total area A, fully submerged in a static mass of liquid. Let the specific weight of the liquid be γ. The surface is immersed in the liquid such that it is inclined making an angle θ with the horizontal. The projection of the surface along its plane intersects with the free surface of liquid (at '*O*' in the figure). The line passing through '*O*' perpendicular to the plane of paper is the axis of intersection of the plane surface projected with the free surface of liquid.

Let $\bar{x}$ be the vertical depth of the centroid of the plane surface below the free surface of liquid. Let the distance of the centroid of the plane surface from the axis of intersection through '*O*' measured along the plane of the inclined surface be $\bar{y}$.

Let us consider a small strip of area dA on the plane surface. Let this strip be located at a vertical depth x below the free surface. Let its location along the plane of the inclined surface from the axis of intersection through '*O*' be y. As the area of the strip is very small, it is assumed that the pressure intensity over the entire strip is uniform (not varying). The magnitude of pressure intensity on the elementary strip of area dA is given by $p = \gamma x$. Hence, total pressure on the strip is given by

dP = uniform pressure intensity over the strip x area of the strip
$= p \times dA = (\gamma x) \times dA$

As $x = y \sin \theta$

$$dP = (\gamma . y \sin \theta) \times dA$$

The total pressure on the entire plane surface of area A can be obtained by dividing the entire area into a number of elementary strips each of area dA and evaluating the total pressures on every such strip and summing them. This can be mathematically represented as

$$P = \int dP = \int \gamma (y \sin \theta) dA = \gamma \sin \theta \int y.dA$$

What $\int y.dA$ in the above expression represents? It represents the sum of first moments of areas of the elementary strips about the axis of intersection passing through '*O*'. By definition, this is equal to the product of the area A of the inclined plane surface and the inclined distance $\bar{y}$ of the centroid of the area from the axis of intersection through '*O*'. That is,

$$\int y.dA = A\bar{y}$$

Putting $\int y.dA = A\bar{y}$ in the expression for P, we have,

$P = \gamma \sin\,\theta (A\bar{y})$

From Figure, $\sin\,\theta = \dfrac{\bar{x}}{\bar{y}}$

Hence, $\bar{x} = \bar{y} \sin\,\theta$

Therefore, $P = \gamma A\bar{x} = (\gamma\bar{x})A$

where, $(\gamma\bar{x})$ = pressure intensity at a vertical depth $\bar{x}$ below the free surface of the liquid (or) pressure intensity at the centroid of the area

It is found that the expression for the magnitude of total pressure is the same irrespective of whether the plane surface is fully submerged in the static mass of liquid in the vertical position or in the inclined position. That is, the total pressure is the product of the pressure intensity at the centroid of the plane surface and the area of the plane surface.

Determination of Centre of Pressure for an Inclined Surface

Let the centre of pressure for the inclined surface be located at a vertical depth $\bar{h}$ below the free liquid surface. Let the location of centre of pressure be at an inclined distance y_p along the plane of the surface from the axis of intersection through 'O'.

Total pressure on the elementary strip of area *dA* shown in figure above is

$dP = (\gamma.\, y \sin\,\theta) \text{ x } dA$

Moment caused by *dP* about the axis of intersection through '*O*' is given by

$(dP)\, y = \{(\gamma.\, y \sin\,\theta) \text{ x } dA\}\, y = \gamma \sin\,\theta\, y^2\, dA$

By considering such elementary strips throughout the area *A* of the inclined plane surface and summing up the moments caused by the total pressure on each of these strips about the axis of intersection through '*O*', we get, by the "Principle of moments",

$Py_p = \int \gamma \sin\,\theta . y^2 dA = \gamma \sin\,\theta \int y^2 dA$

In the above equation, the quantity $\int y^2 dA$ represents the sum of second moments of areas of the elementary strips about the axis of intersection through '*O*'. By definition, the sum of second moment of areas of elementary strips about the axis through 'O' represents the moment of inertia of the plane surface I_o about the axis through 'O'. That is,

$$I_O = \int y^2 . dA$$

Putting $\int y^2 dA = I_O$ in the expression, $P y_p = \gamma \sin\theta \int y^2 dA$, we have,

$$P y_p = \gamma \sin\theta . I_O$$

$$\Rightarrow y_p = \frac{(\gamma \sin\theta) I_O}{P}$$

Further from the "Parallel axes theorem" for moment of inertia, we have,

$$I_o = I_G + A \bar{y}^2$$

where I_G is the moment of inertia of the inclined plane surface about the axis passing through the centroid of the area of the plane surface. It should be noted that the axis through the centroid of the plane surface is perpendicular to the plane of the paper and parallel to the axis of intersection through the point 'O'.

Putting $I_o = I_G + A\bar{y}^2$ in the expression $y_p = \frac{(\gamma \sin\theta) I_O}{P}$, we have,

$$y_p = \frac{(\gamma \sin\theta)(I_G + A\bar{y}^2)}{P}$$

Putting $P = \gamma A (\bar{y} \sin\theta)$ in the above expression for y_p

$$y_p = \frac{(\gamma \sin\theta)(I_G + A\bar{y}^2)}{\gamma A(\bar{y}\sin\theta)} = \frac{\gamma \sin\theta . I_G}{\gamma A(\bar{y}\sin\theta)} + \frac{\gamma \sin\theta . A\bar{y}^2}{\gamma A(\bar{y}\sin\theta)} = \frac{I_G}{A\bar{y}} + \bar{y}$$

But from figure, we have,

$$y_p = \frac{\bar{h}}{\sin\theta} \text{ and } \bar{y} = \frac{\bar{x}}{\sin\theta}$$

Hence, the expression, $y_p = \frac{I_G}{A\bar{y}} + \bar{y}$ becomes

$$\frac{\bar{h}}{\sin\theta} = \frac{I_G}{A\left(\dfrac{\bar{x}}{\sin\theta}\right)} + \left(\frac{\bar{x}}{\sin\theta}\right)$$

$$\Rightarrow \frac{\bar{h}}{\sin\theta} = \frac{I_G \sin\theta}{A\bar{x}} + \left(\frac{\bar{x}}{\sin\theta}\right)$$

$$\Rightarrow \bar{h} = \left\{\frac{I_G \sin\theta}{A\bar{x}} + \left(\frac{\bar{x}}{\sin\theta}\right)\right\}\sin\theta = \frac{I_G \sin^2\theta}{A\bar{x}} + \bar{x}$$

$$\Rightarrow \bar{h} = \bar{x} + \frac{I_G \sin^2\theta}{A\bar{x}}$$

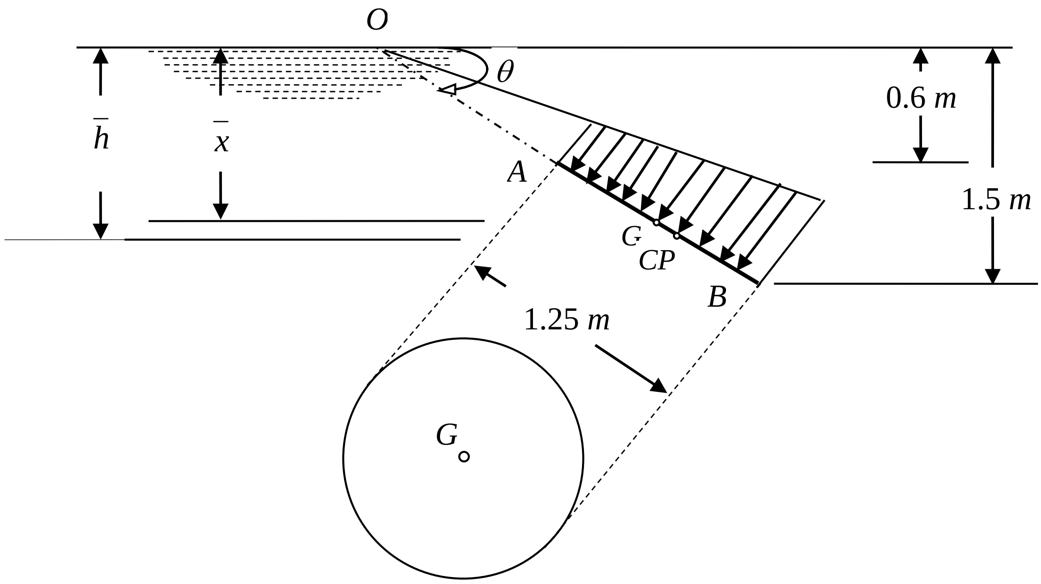
The above equation gives the vertical depth of centre of pressure below the free surface of liquid, for an inclined plane surface wholly submerged in a static mass of liquid. When $\theta = 90°$, that is, when the inclination of the plane surface is 90° with the horizontal, it becomes the case of a vertical plane surface immersed in a static mass of liquid. Putting $\theta = 90°$ in the above expression for $\bar{h}$, we have,

$$\bar{h} = \bar{x} + \frac{I_G \sin^2 90^\circ}{A\bar{x}} = \bar{x} + \frac{I_G}{A\bar{x}}$$

This expression is the same as that derived for the vertical depth of centre of pressure below the free surface of liquid for a vertical plane surface wholly submerged in a static mass of liquid.

Example 4. A circular lamina 1.25 *m* in diameter is immersed in water so that the distance of its edge measured vertically below the free surface varies from 0.6 *m* to 1.5 *m*. Find the total force due to the water acting on one side of the lamina, and the vertical distance of the centre of pressure below the surface.

Solution.



Pressure intensity at the upper edge of the plate, $p_A = \gamma h_A = 9810\ N\,m^{-3} \times 0.6\ m$
$= 5886\ N\,m^{-2}$

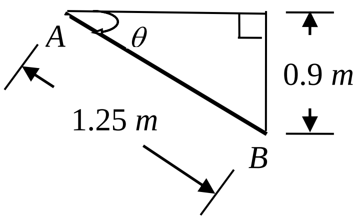
Pressure intensity at the lower edge of the plate, $p_B = \gamma h_B = 9810\ N\,m^{-3} \times 1.5\ m$
$= 14715\ N\,m^{-2}$

Total pressure on the plate = (average pressure intensity over the plate) x (area of the plate)

$$= \left(\frac{p_A + p_B}{2}\right)\left(\frac{\pi}{4} \text{ x } D^2\right) = \left(\frac{5886 + 14715}{2}\right)\left(\frac{\pi}{4} \text{ x } 1.25^2\right)$$

$$= 12646\ N = 12.646\ kN$$

Let the projection of the plate meets the free water surface at O and make an angle θ with the free surface. It should be noted that the plate makes an angle θ with the imaginary horizontal line passing through the upper edge of the plate A.



$$\sin\theta = \frac{0.9}{1.25}$$

$$\Rightarrow \theta = 46.05$$

Location of centroid of the plate, G:

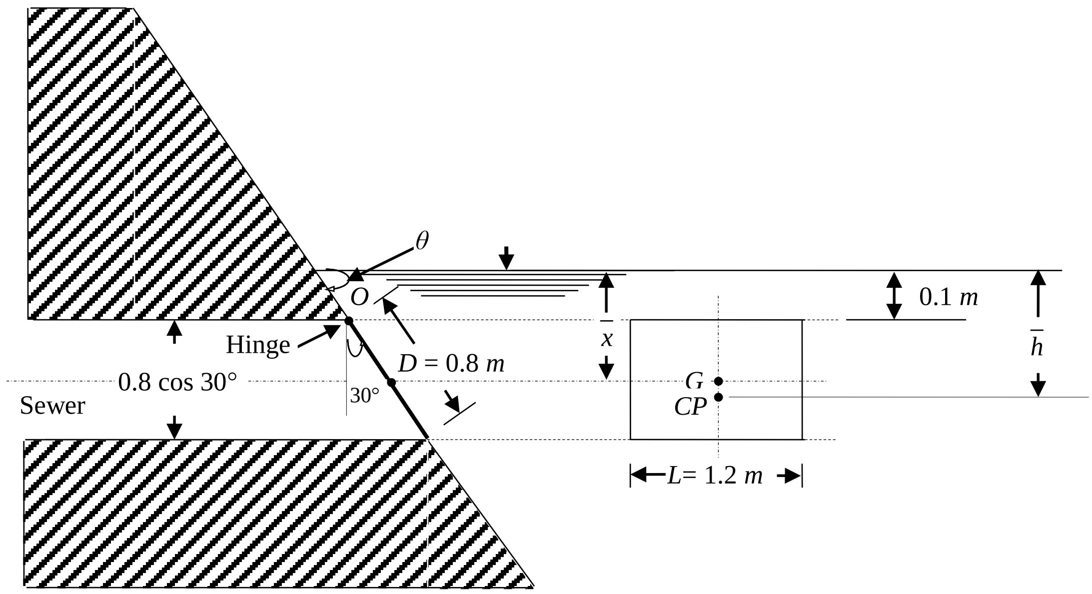
$$\bar{x} = 0.6 + \left(\frac{1.5 - 0.6}{2}\right) = 1.05\ m$$

Location of centre of pressure CP:

$$\bar{h} = \bar{x} + \frac{I_G \sin^2\theta}{A\bar{x}}$$

$$\bar{h} = 1.05 + \frac{\left(\dfrac{\pi \text{ x } 1.25^4}{64}\right) \text{x} \sin^2 46.05^\circ}{1.2277 \text{ x } 1.05} = 1.05 + 0.048 = 1.098\ m$$

Example 5. A gate at the end of a sewer measures 0.8 *m* by 1.2 *m* wide. It is hinged along its top edge and hangs at an angle of 30° to the vertical, this being the angle of the banks of a trapezoidal river channel. (a) Calculate the hydrostatic force on the gate and the vertical distance between the centroid of the gate, *G*, and the centre of pressure, *CP*, when the river level is 0.1 *m* above the top of the hinge.



Figure

Depth of rectangular gate along the bank of the trapezoidal river channel, *D* = 0.8 *m*.

Width of rectangular gate perpendicular to the plane of the paper, *L* = 1.2 *m*

Area of rectangular sluice gate in contact with water, *A* = (width of gate, *L*) x (depth of gate, *D*)
= 1.2 *m* x 0.8 *m*
= 0.96 m^2

The upper edge of the plate (perpendicular to the plane of paper at hinge *O*) is at a depth 0.1 *m* below the free surface of water. The gate is inclined at 30° to the vertical. '*G*' is the centre of gravity of the gate. '*G*' is located at a distance 0.4 m along the gate below the hinge *O*. The vertical location of *G* below *O* is (0.8 cos 30°) equal to 0.693 *m*.

Hence, the vertical location of *G* below the free surface of water,

$\bar{x}$ = 0.1 + 0.4 cos 30° = 0.1 + 0.4 (0.866) = 0.446 *m*

Total pressure exerted by water on the gate,

$P = \gamma A\bar{x}$ = (9810 $N\,m^{-3}$) x (0.96 m^2) x (0.446 *m*) = 4200 *N*
P acts normal to the gate at the centre of pressure.

Location of centre of pressure CP:

$$\bar{h} = \bar{x} + \frac{I_G \sin^2\theta}{A\bar{x}}$$

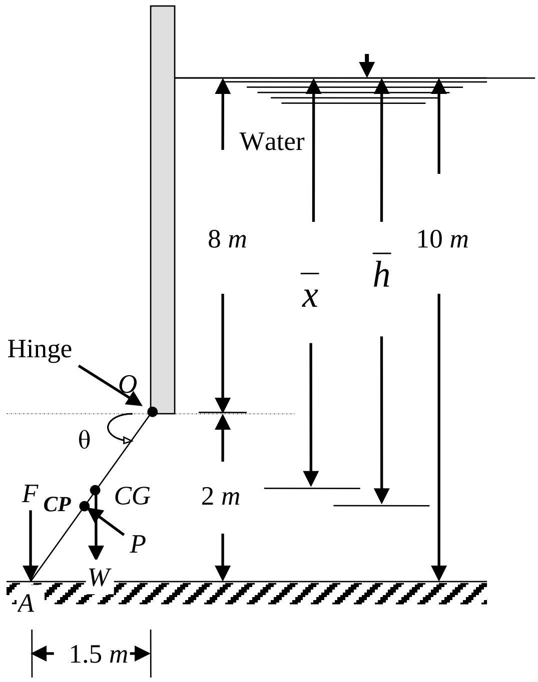
where θ is the angle which the projection of the top edge of the gate makes with the free water surface in the river.

$$\bar{h} = 0.446 + \frac{\left(\frac{1.2 \text{ x } 0.8^3}{12}\right) \text{x} \sin^2 60^\circ}{0.96 \text{ x } 0.446} = 0.446 + 0.090 = 0.455\ m$$

Hence, the vertical distance between the centre of pressure, *CP*, and the centre of gravity of the gate, *G*, is given by

$\bar{h} - \bar{x}$ = 0.455 – 0.446 = 0.090 *m*

Example 6. In the figure shown, determine the force '*F*' to close the gate, if the mass of the gate is 1000 *kg*. Width of gate = 3 *m*



Mass of the gate = 1000 *kg*

Weight of the gate = (mass of the gate) x (acceleration due to gravity)

$= (1000\ kg) \times (9.81\ m\ s^{-2})$

$W = 9810\ N$

Width of gate perpendicular to the plane of paper = 3 *m*

Let θ be the inclination of the gate with the horizontal.

$$\tan\theta = \frac{2}{1.5} = 1.333$$

$\Rightarrow \theta = \tan^{-1}(1.333) = 53.1^\circ$

Dimension $\overline{OA}$ of rectangular gate = 2 / sin θ = 2 / sin (53.1°) = 2 / 0.8 = 2.5 *m*

Area of gate in contact with water, $A = 3\ m \times 2.5\ \text{m} = 7.5\ m^2$

The centroid of the gate, *CG*, is located at a vertical depth 1 *m* below the hinge *O*. Hence, the vertical depth of the centroid, *CG*, of the gate below the free surface of water, $\bar{x} = 8 + 1 = 9\ m$

Total pressure acting on the inclined gate, $P = \gamma A \bar{x}$

$= \left(9810\ N\ m^{-3}\right) \times \left(7.5\ m^2\right) \times (9\ m)$

$= 662175\ N$

Moment of inertia of the rectangular gate of dimensions 3 m x 2.5 m about the centroidal axis is given by

$$I_G = \frac{3\,m \text{ x } (2.5\,m)^3}{12} = 3.90\ m^4$$

The vertical depth of the centre of pressure, CP, below the free surface of water is given by

$$\bar{h} = \bar{x} + \frac{I_G \sin^2\theta}{A\bar{x}}$$
$$= 9\,m + \frac{(3.90\,m^4)\sin^2(53.1)}{(4.8\,m) \text{ x } (9\,m)} = 9.036\ m$$

For static equilibrium of the gate, the algebraic sum of moments caused by forces acting on the gate about the hinge O must be equal to zero. That is,

P x (perpendicular distance between the line of action of P and the hinge O) –
W x (perpendicular distance between the line of action of W and the hinge O) –
F x (perpendicular distance between the line of action of F and the hinge O) = 0

Perpendicular distance between the line of action of P and the hinge O
= distance along the edge of the gate between the centre of pressure 'CP' and the hinge ''
= distance along the edge of the gate between the centre of pressure 'CP' and the centre of gravity 'CG' of the gate + distance along the edge of the gate between the centre of gravity 'CG' of the gate and the hinge 'O'

$$= \frac{\bar{h} - \bar{x}}{\sin\theta} + \frac{2.5\,m}{2} = \frac{9.036 - 9.0}{\sin 53.1^\circ} + 1.25\,m = 0.045\ m + 1.25\ m = 1.295\ m$$

Perpendicular distance between the line of action of W and the hinge O = 1.5 / 2 = 0.75 m

$\Rightarrow P(1.295\,m) - W(0.75\,m) - F(1.5\,m) = 0$
$\Rightarrow$ (662175 N x 1.295 m) – (9810 N x 0.75 m) – (F x 1.5 m) = 0
$\Rightarrow F = 566773\ N$

Hence, force required to close the gate, $F = 566773\ N$