Fundamentals of Neural Networks

Fundamentals of Neural Networks

Soft Computing

Neural network, topics : Introduction, biological neuron model, artificial neuron model, neuron equation. Artificial neuron : basic elements, activation and threshold function, piecewise linear and sigmoidal function. Neural network architectures : single layer feed- forward network, multi layer feed-forward network, recurrent networks. Learning methods in neural networks : unsupervised Learning - Hebbian learning, competitive learning; Supervised learning - stochastic learning, gradient descent learning; Reinforced learning. Taxonomy of neural network systems : popular neural network systems, classification of neural network systems as per learning methods and architecture. Single-layer NN system : single layer perceptron, learning algorithm for training perceptron, linearly separable task, XOR problem, ADAptive LINear Element (ADALINE) - architecture, and training. Applications of neural networks: clustering, classification, pattern recognition, function approximation, prediction systems.

Fundamentals of Neural Networks

Soft Computing

Topics

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What is Neural Net ?

- A neural net is an artificial representation of the human brain that tries to simulate its learning process. An artificial neural network (ANN) is often called a "Neural Network" or simply Neural Net (NN).
- Traditionally, the word neural network is referred to a network of biological neurons in the nervous system that process and transmit information.
- Artificial neural network is an interconnected group of artificial neurons that uses a mathematical model or computational model for information processing based on a connectionist approach to computation.
- The artificial neural networks are made of interconnecting artificial neurons which may share some properties of biological neural networks.
- Artificial Neural network is a network of simple processing elements (neurons) which can exhibit complex global behavior, determined by the connections between the processing elements and element parameters.

1. Introduction

Neural Computers mimic certain processing capabilities of the human brain.

- Neural Computing is an information processing paradigm, inspired by biological system, composed of a large number of highly interconnected processing elements (neurons) working in unison to solve specific problems.
- Artificial Neural Networks (ANNs), like people, learn by example.
- An ANN is configured for a specific application, such as pattern recognition or data classification, through a learning process.
- Learning in biological systems involves adjustments to the synaptic connections that exist between the neurons. This is true of ANNs as well.

Why Neural Network

Neural Networks follow a different paradigm for computing.

- The conventional computers are good for - fast arithmetic and does what programmer programs, ask them to do.
- The conventional computers are not so good for - interacting with noisy data or data from the environment, massive parallelism, fault tolerance, and adapting to circumstances.
- The neural network systems help where we can not formulate an algorithmic solution or where we can get lots of examples of the behavior we require.
- Neural Networks follow different paradigm forcomputing.

  The von Neumann machines are based on the processing/memory abstraction of human information processing.

  The neural networks are based on the parallel architecture of biological brains.
- Neural networks are a form of multiprocessor computer system, with
  - simple processing elements ,
  - a high degree of interconnection,
  - simple scalar messages, and
  - adaptive interaction between elements.

Research History

The history is relevant because for nearly two decades the future of Neural network remained uncertain.

McCulloch and Pitts (1943) are generally recognized as the designers of the first neural network. They combined many simple processing units together that could lead to an overall increase in computational power. They suggested many ideas like : a neuron has a threshold level and once that level is reached the neuron fires. It is still the fundamental way in which ANNs operate. The McCulloch and Pitts's network had a fixed set of weights.

Hebb (1949) developed the first learning rule, that is if two neurons are active at the same time then the strength between them should be increased.

In the 1950 and 60's, many researchers (Block, Minsky, Papert, and Rosenblatt worked on perceptron. The neural network model could be proved to converge to the correct weights, that will solve the problem. The weight adjustment (learning algorithm) used in the perceptron was found more powerful than the learning rules used by Hebb. The perceptron caused great excitement. It was thought to produce programs that could think.

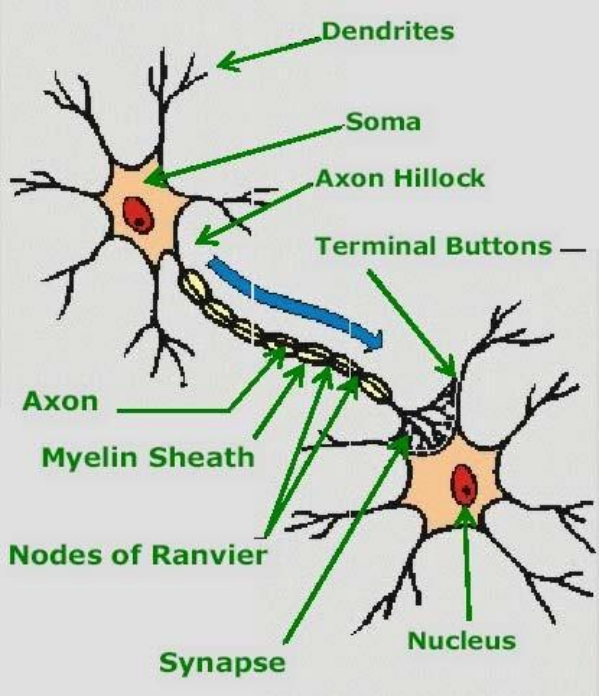
Minsky & Papert (1969) showed that perceptron could not learn those functions which are not linearly separable.

The neural networks research declined throughout the 1970 and until mid 80's because the perceptron could not learn certain important functions.

Neural network regained importance in 1985-86. The researchers, Parker and LeCun discovered a learning algorithm for multi-layer networks called back propagation that could solve problems that were not linearly separable.

Biological Neuron Model

The human brain consists of a large number, more than a billion of neural cells that process information. Each cell works like a simple processor. The massive interaction between all cells and their parallel processing only makes the brain's abilities possible.



Dendrites are branching fibers that extend from the cell body or soma.

Soma or cell body of a neuron contains the nucleus and other structures, support chemical processing and production of neurotransmitters.

Axon is a singular fiber carries information away from the soma to the synaptic sites of other neurons (dendrites and somas), muscles, or glands.

Axon hillock is the site of summation for incoming information. At any moment, the collective influence of all neurons that conduct impulses to a given neuron will determine whether or not an

Fig. Structure of Neuron

action potential will be initiated at the axon hillock and propagated along the axon.

Myelin Sheath consists of fat-containing cells that insulate the axon from electrical activity. This insulation acts to increase the rate of transmission of signals. A gap exists between each myelin sheath cell along the axon. Since fat inhibits the propagation of electricity, the signals jump from one gap to the next.

Nodes of Ranvier are the gaps (about 1 □m) between myelin sheath cells long axons are Since fat serves as a good insulator, the myelin sheaths speed the rate of transmission of an electrical impulse along the axon.

Synapse is the point of connection between two neurons or a neuron and a muscle or a gland. Electrochemical communication between neurons takes place at these junctions.

Terminal Buttons of a neuron are the small knobs at the end of an axon that release chemicals called neurotransmitters.

- **Information flow in a Neural Cell**

The input /output and the propagation of information are shown below.

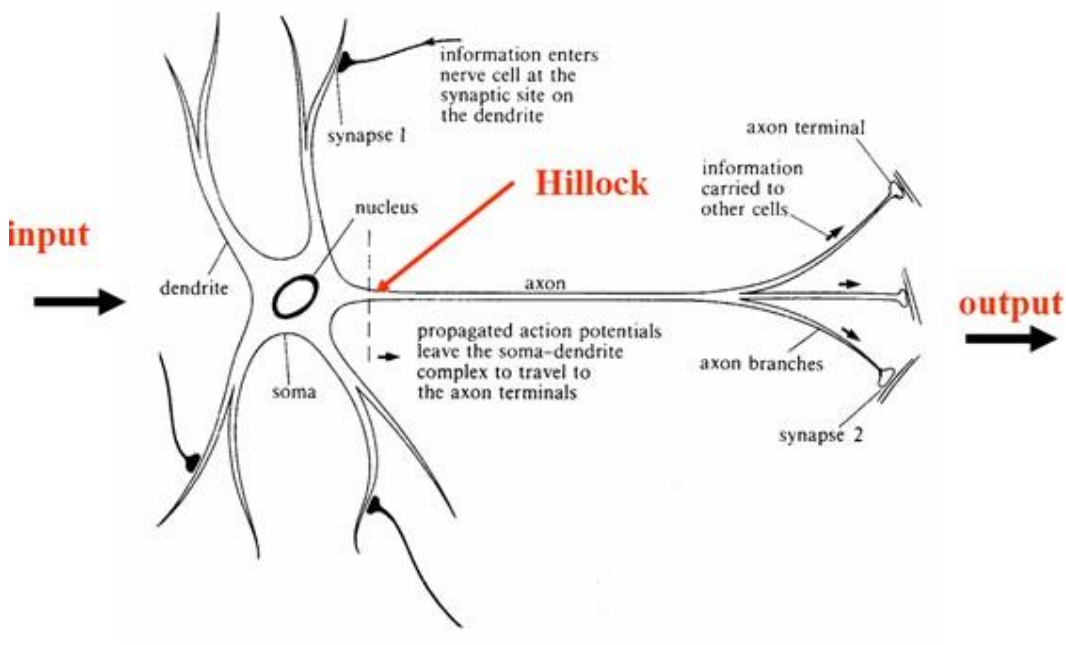


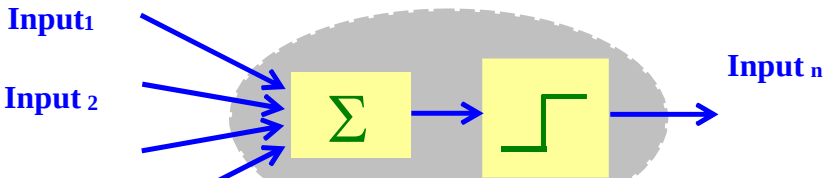
Fig. Structure of a neural cell in the human brain

- Dendrites receive activation from other neurons.
- Soma processes the incoming activations and converts them into output activations.
- Axons act as transmission lines to send activation to other neurons.
- Synapses the junctions allow signal transmission between the axons and dendrites.
- The process of transmission is by diffusion of chemicals called neuro-transmitters.

McCulloch-Pitts introduced a simplified model of this real neurons. Artificial Neuron Model

- **The McCulloch-Pitts Neuron**

This is a simplified model of real neurons, known as a Threshold Logic Unit.



Output

- A set of input connections brings in activations from other neurons.
- A processing unit sums the inputs, and then applies a non-linear activation function (i.e. squashing / transfer / threshold function).
- An output line transmits the result to other neurons.

In other words ,

- The input to a neuron arrives in the form of signals.
- The signals build up in the cell.
- Finally the cell discharges (cell fires) through the output .
- The cell can start building up signals again.

Notations

Recaps : Scalar, Vectors, Matrices and Functions

Scalar : The number x_i can be added up to give a scalar number.

$$s = x_1 + x_2 + x_3 + \ldots + x_n = \sum_{i=1}^{n} x_i$$

Vectors : An ordered sets of related numbers. Row Vectors **(1 x n)**

$$X = (x_1 , x_2 , x_3 , \ldots , x_n) , \qquad Y = (y_1 , y_2 , y_3 , \ldots , y_n)$$

Add : Two vectors of same length added to give another vector.

$$Z = X + Y = (x_1 + y_1 , x_2 + y_2 , \ldots\ldots\ldots\ldots , x_n + y_n)$$

Multiply: Two vectors of same length multiplied to give a scalar.

$$p = X . Y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Matrices : m x n matrix , row no = m , column no = n

$$W = \begin{Bmatrix} w11 & w11 & . . & . . & w1n \\ w21 & w21 & . . & . . & w21 \\ . & . & . . & . . & . \\ . & . & . . & . . & . \\ wm1 & w11 & \ldots\ldots\ldots\ldots & & wmn \end{Bmatrix}$$

Add or Subtract : Matrices of the same size are added or subtracted component by component. **A + B = C** , **cij = aij + bij**

$$\begin{Bmatrix} a11 & a12 \\ a21 & a22 \end{Bmatrix} + \begin{Bmatrix} b11 & b12 \\ b21 & b22 \end{Bmatrix} = \begin{Bmatrix} c11 = a11+b11 & c12 = a12+b12 \\ C21 = a21+b21 & C22 = a22+b22 \end{Bmatrix}$$

Multiply : matrix **A** (m x n) multiplied by matrix **B** (n x p) gives matrix **C** (m x p).

elements $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$\begin{Bmatrix} a11 & a12 \\ a21 & a22 \end{Bmatrix} x \begin{Bmatrix} b11 & b12 \\ b21 & b22 \end{Bmatrix} = \begin{Bmatrix} c11 & c12 \\ c21 & c22 \end{Bmatrix}$$

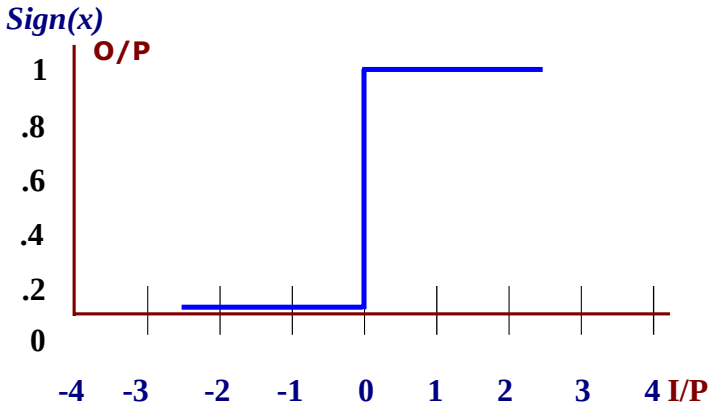
c11 = (a11 x b11) + (a12 x B21) c12 = (a11 x b12) + (a12 x B22) C21 = (a21 x b11) + (a22 x B21) C22 = (a21 x b12) + (a22 x B22)

Functions

The Function **y= f(x)** describes a relationship, an input-output mapping, from **x** to **y**.

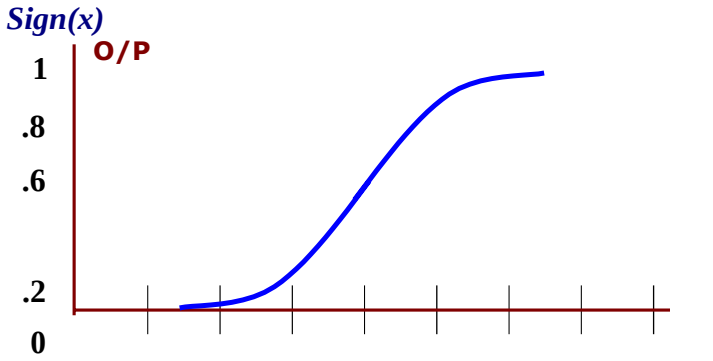
- **Threshold or Sign function :** **sgn(x)** defined as

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



- **Threshold or Sign function :** **sigmoid(x)** defined as a smoothed (differentiable) form of the threshold function

$$\text{sigmoid}(x) = \frac{1}{1 + e^{-x}}$$



-4 -3 -2 -1 0 1 2 3 4 I/P

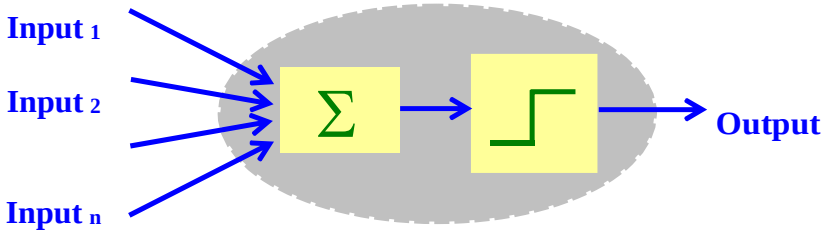
2. Model of Artificial Neuron

A very simplified model of real neurons is known as a Threshold Logic Unit (TLU). The model is said to have:

- A set of synapses (connections) brings in activations from other neurons.
- A processing unit sums the inputs, and then applies a non-linear activation function (i.e. squashing / transfer / thresholdfunction).
- An output line transmits the result to other neurons.

McCulloch-Pitts (M-P) Neuron Equation

McCulloch-Pitts neuron is a simplified model of real biological neuron.



Simplified Model of Real Neuron

(Threshold Logic Unit)

The equation for the output of a McCulloch-Pitts neuron as a function of **1** to **n** inputs is written as

$$\text{Output} = \text{sgn}\left(\sum_{i=1}^{n} \text{Input } i - \theta\right)$$

where θ is the neuron's activation threshold.

If $\sum_{i=1}^{n} \text{Input } i \geq \theta$ then Output = 1

If $\sum_{i=1}^{n} \text{Input } i < \theta$ then Output = 0

In this McCulloch-Pitts neuron model, the missing features are :

- Non-binary input and output,
- Non-linear summation,
- Smooth thresholding,
- Stochastic, and
- Temporal information processing.

Artificial Neuron - Basic Elements

Neuron consists of three basic components - weights, thresholds, and a single activation function.

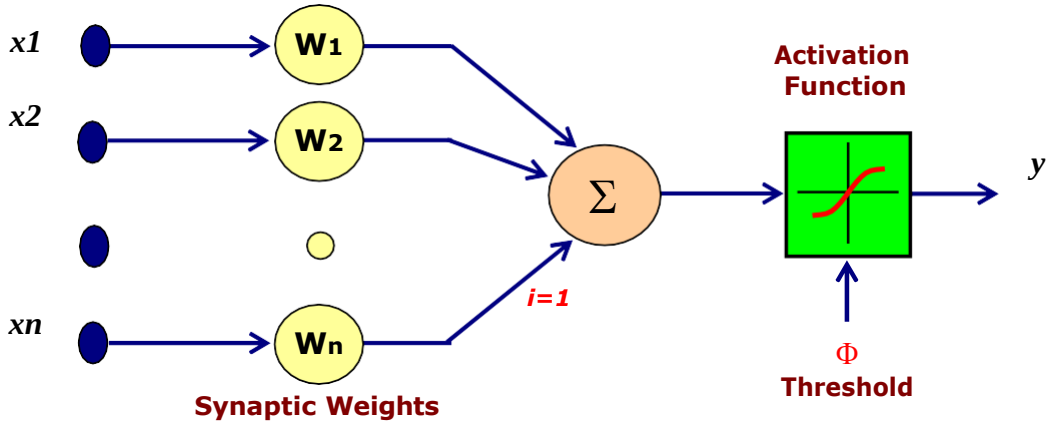


Fig Basic Elements of an Artificial Linear Neuron

- **Weighting Factors w**

  The values **w1 , w2 , . . . wn** are weights to determine the strength of input vector **X = [x1 , x2 , . . . , xn]^T**. Each input is multiplied by the associated weight of the neuron connection **X^T W**. The +ve weight excites and the -ve weight inhibits the nodeoutput.

  $$I = X^T.W = x_1 w_1 + x_2 w_2 + \ldots + x_n w_n = \sum_{i=1}^{n} x_i w_i$$

- **Threshold □**

  The node's internal threshold □ is the magnitude offset. It affects the activation of the node output **y** as:

  $$Y = f(I) = f\{ \sum_{i=1}^{n} x_i w_i - \square k \}$$

To generate the final output **Y** , the sum is passed on to a non-linear filter **f** called Activation Function or Transfer function or Squash function which releases the output **Y**.

- **Threshold for a Neuron**

In practice, neurons generally do not fire (produce an output) unless their total input goes above a threshold value.

The total input for each neuron is the sum of the weighted inputs to the neuron minus its threshold value. This is then passed through the sigmoid function. The equation for the transition in a neuron is :

a = 1/(1 + exp(- x)) where

x = $\sum_i$ ai wi - Q

a is the activation for the neuron **ai** is the activation for neuron ***i*** **wi** is the weight

Q is the threshold subtracted

- **Activation Function**

An activation function **f** performs a mathematical operation on the signal output. The most common activation functions are:

- Linear Function,
- Piecewise Linear Function,
- Tangent hyperbolic function
- Threshold Function,
- Sigmoidal (S shaped) function,

The activation functions are chosen depending upon the type of problem to be solved by the network.

Activation Functions f - Types

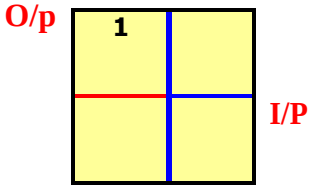
Over the years, researches tried several functions to convert the input into an outputs. The most commonly used functions are described below.

- **I/P** Horizontal axis shows sum of inputs .
- **O/P** Vertical axis shows the value the function produces ie output.
- All functions **f** are designed to produce values between **0** and **1**.

- **Threshold Function**

A threshold (hard-limiter) activation function is either a binary type or a bipolar type as shown below.

binary threshold Output of a binary threshold function produces :

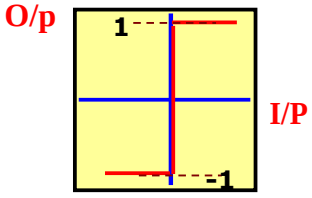


1 if the weighted sum of the inputs is +ve,

0 if the weighted sum of the inputs is -ve.

$$Y = f(I) = \begin{cases} 1 & \text{if } I \geq 0 \\ 0 & \text{if } I < 0 \end{cases}$$

bipolar threshold Output of a bipolar threshold function produces :



1 if the weighted sum of the inputs is +ve,

-1 if the weighted sum of the inputs is -ve.

$$Y = f(I) = \begin{cases} 1 & \text{if } I \geq 0 \\ & \end{cases}$$

Neuron with hard limiter activation function is called McCulloch-Pitts model.

- **Piecewise Linear Function**

This activation function is also called saturating linear function and can have either a binary or bipolar range for the saturation limits of the output. The mathematical model for a symmetric saturation function is described below.

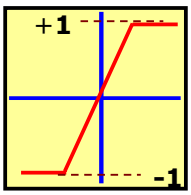
Piecewise Linear This is a sloping function that produces:

-1 for a -ve weighted sum of inputs,

1 for a +ve weighted sum of inputs.

□ **I** proportional to input for values between **+1** and **-1** weighted sum,

O/p



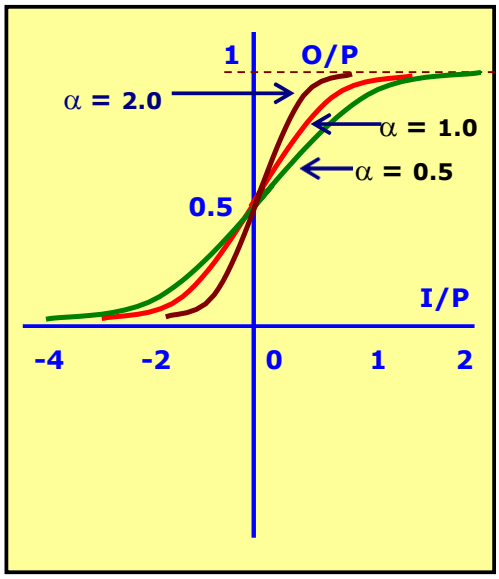
I/P

$$Y = f(I) = \begin{cases} 1 & \text{if } I \geq 0 \\ I & \text{if } -1 \leq I \leq 1 \\ -1 & \text{if } I \leq 0 \end{cases}$$

- **Sigmoidal Function** (S-shape function)

The nonlinear curved S-shape function is called the sigmoid function. This is most common type of activation used to construct the neural networks. It is mathematically well behaved, differentiable and strictly increasing function.

Sigmoidal function



The sigmoidal function is achieved using exponential equation.

A sigmoidal transfer function can be written in the form:

$$Y = f(I) = \frac{1}{1 + e^{-\lambda I}}, \quad 0 \leq f(I) \leq 1$$

$$= 1/(1 + \exp(-\lambda I)), \quad 0 \leq f(I) \leq 1$$

This is explained as

0 for large -ve input values,

1 for large +ve values, with a smooth transition between thetwo.

λ is slope parameter also called shape parameter; symbol the λ is also used to represented this parameter.

By varying different shapes of the function can be obtained which adjusts the abruptness of the function as it changes between the two asymptotic values.

- **Example :**

The neuron shown consists of four inputs with the weights.

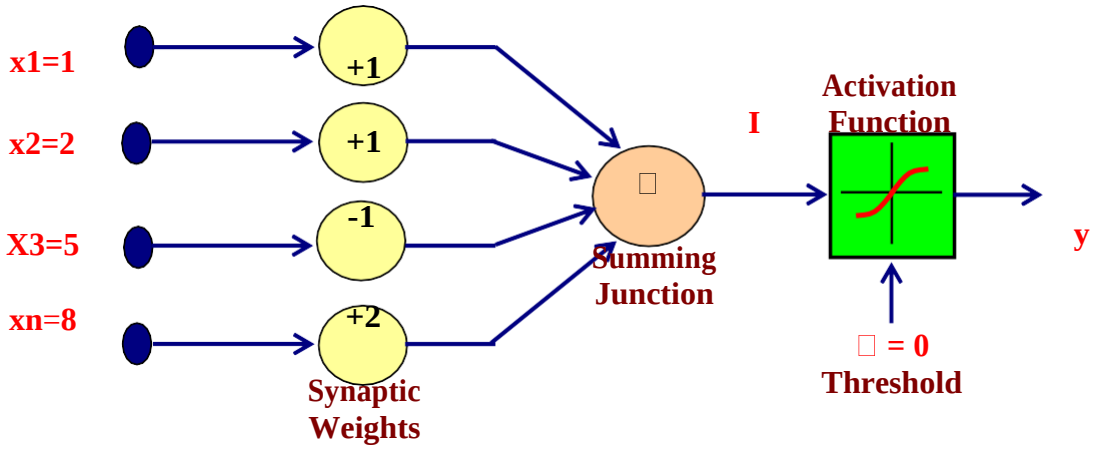


Fig Neuron Structure of Example

The output **I** of the network, prior to the activation function stage, is

$$I = X^T . W = \begin{bmatrix} 1 & 2 & 5 & 8 \end{bmatrix} \bullet \begin{pmatrix} +1 \\ +1 \\ -1 \\ +2 \end{pmatrix} = 14$$

$$= (1 \times 1) + (2 \times 1) + (5 \times -1) + (8 \times 2) = 14$$

With a binary activation function the outputs of the neuron is:

y (threshold) = 1;

3. Neural Network Architectures

An Artificial Neural Network (ANN) is a data processing system, consisting large number of simple highly interconnected processing elements as artificial neuron in a network structure that can be represented using a directed graph **G**, an ordered 2-tuple **(V, E)** , consisting a set **V** of vertices and a set **E** of edges.

- The vertices may represent neurons (input/output) and
- The edges may represent synaptic links labeled by the weights attached. Example :

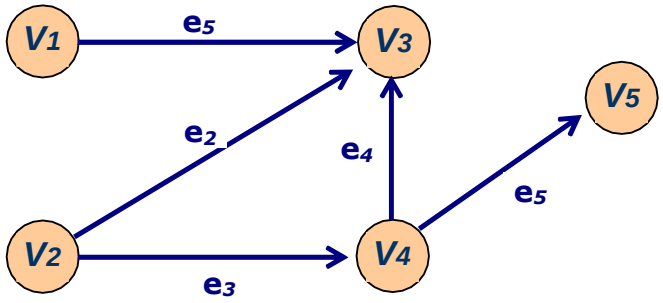


Fig. Directed Graph

Vertices V = { v_1 , v_2 , v_3 , v_4, v_5 } Edges

E = { e_1 , e_2 , e_3 , e_4, e_5 }

Single Layer Feed-forward Network

The Single Layer Feed-forward Network consists of a single layer of weights , where the inputs are directly connected to the outputs, via a series of weights. The synaptic links carrying weights connect every input to every output , but not other way. This way it is considered a network of **feed-forward** type. The sum of the products of the weights and the inputs is calculated in each neuron node, and if the value is above some threshold (typically **0**) the neuron fires and takes the activated value (typically **1**); otherwise it takes the deactivated value (typically **-1**).

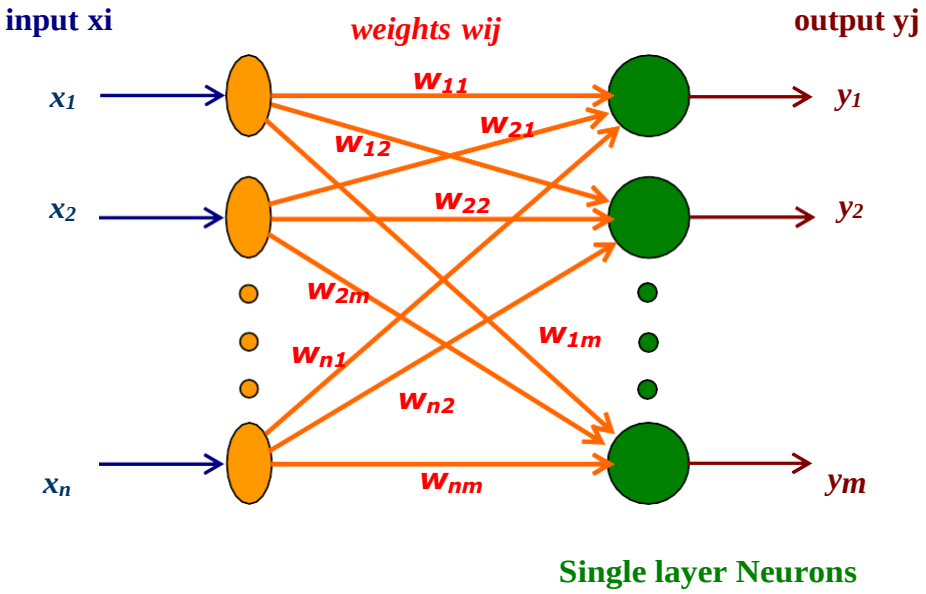


Fig. Single Layer Feed-forward Network

Multi Layer Feed-forward Network

The name suggests, it consists of multiple layers. The architecture of this class of network, besides having the input and the output layers, also have one or more intermediary layers called hidden layers. The computational units of the hidden layer are known as hidden neurons.

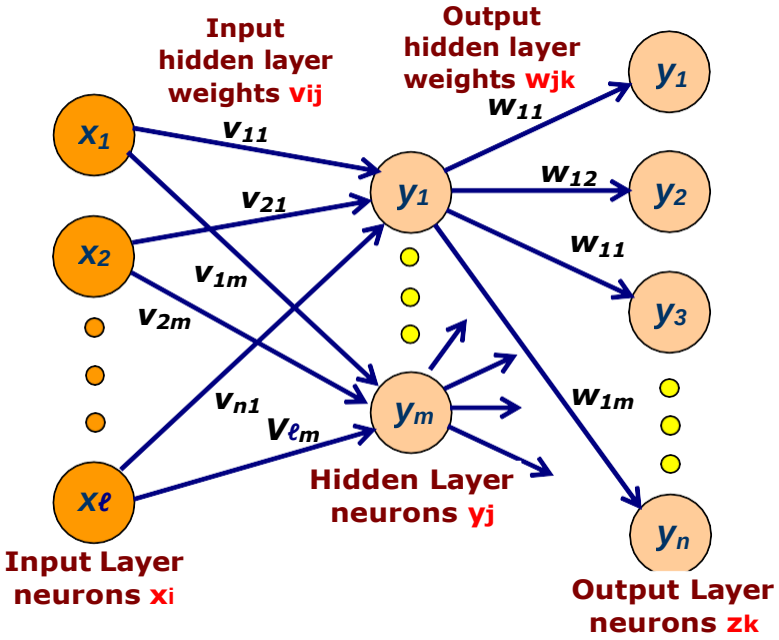


Fig. Multilayer feed-forward network in *(ℓ – m – n)* configuration.

- The hidden layer does intermediate computation before directing the input to output layer.
- The input layer neurons are linked to the hidden layer neurons; the weights on these links are referred to as **input-hidden layer weights**.
- The hidden layer neurons and the corresponding weights are referred to as **output-hidden layer weights**.
- A multi-layer feed-forward network with ℓ input neurons, m_1 neurons in the first hidden layers, m_2 neurons in the second hidden layers, and n output neurons in the output layers is written as **$(\ell - m_1 - m_2 - n)$.**

The Fig. above illustrates a multilayer feed-forward network with a configuration ***(ℓ - m – n).***

Recurrent Networks

The Recurrent Networks differ from feed-forward architecture. A Recurrent network has at least one feed back loop.

Example :

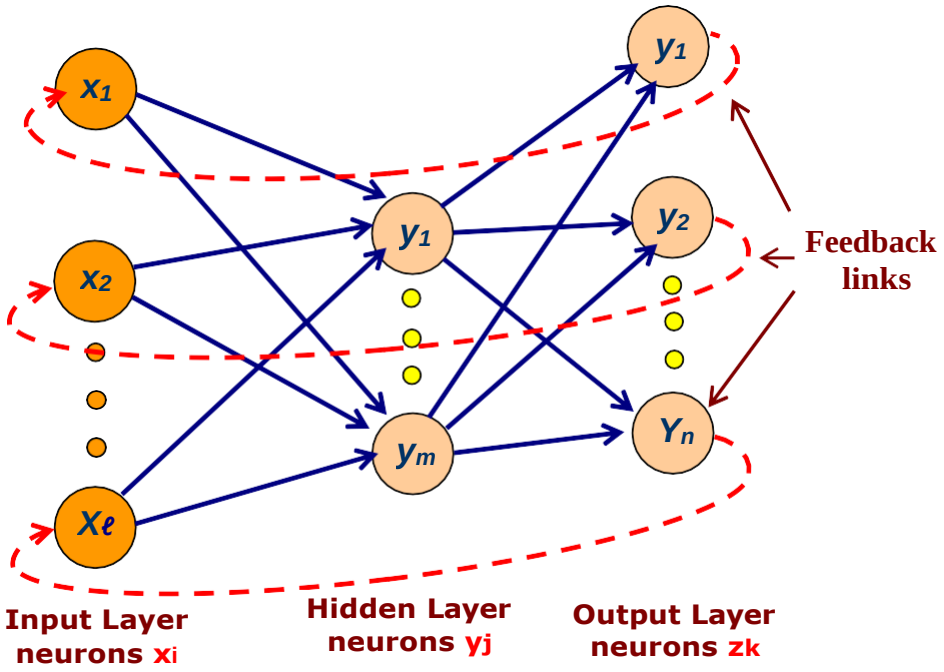


Fig Recurrent Neural Network

There could be neurons with self-feedback links; that is the output of a neuron is fed back into it self as input.

4. Learning Methods in Neural Networks

The learning methods in neural networks are classified into three basic types :

- Supervised Learning,
- Unsupervised Learning and
- Reinforced Learning

These three types are classified based on :

- presence or absence of **teacher** and
- the information provided for the system to learn.

These are further categorized, based on the **rules** used, as

- Hebbian,

- Gradient descent,
- Competitive and
- Stochastic learning.

- **Classification of Learning Algorithms**

  Fig. below indicate the hierarchical representation of the algorithms mentioned in the previous slide. These algorithms are explained in subsequent slides.

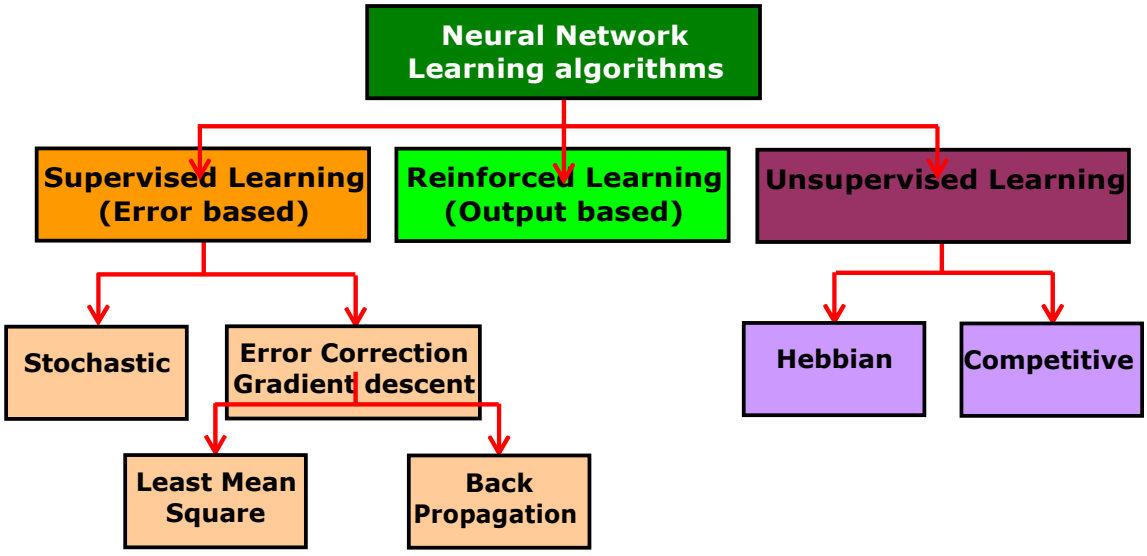


Fig. Classification of learning algorithms

- **Supervised Learning**
  - A teacher is present during learning process and presents expected output.
  - Every input pattern is used to train thenetwork.
  - Learning process is based on comparison, between network's computed output and the correct expected output, generating "error".
  - The "error" generated is used to change network parameters thatresult improved performance.

- **Unsupervised Learning**
  - No teacher is present.
  - The expected or desired output is not presented to the network.
  - The system learns of it own by discovering and adapting to the structural features in the input patterns.

- **Reinforced learning**
  - A teacher is present but does not present the expected or desired output but only indicated if the computed output is correct or incorrect.
  - The information provided helps the network in its learning process.
  - A reward is given for correct answer computed and a penalty for a wrong answer.

Note : The Supervised and Unsupervised learning methods are most popular forms of learning compared to Reinforced learning.

- **Hebbian Learning**

  Hebb proposed a rule based on correlative weight adjustment.

  In this rule, the input-output pattern pairs (X_i , Y_i) are associated by the weight matrix W, known as correlation matrix computed as

$$W = \sum_{i=1}^{n} X_i Y_i^T$$

where Yi^T is the transpose of the associated output vector Yi

There are many variations of this rule proposed by the other researchers (Kosko, Anderson, Lippman) .

- **Gradient descent Learning**

  This is based on the minimization of errors E defined in terms of weights and the activation function of the network.

  - Here, the activation function of the network is required to be differentiable, because the updates of weight is dependent on the gradient of the error E.

  - If ΔWij is the weight update of the link connecting the $i\ th$ and the $j\ th$ neuron of the two neighboring layers, then ΔWij is defined as

    $$\Delta Wij = \eta (\partial E / \partial Wij)$$

    where η is the learning rate parameters and $(\partial E / \partial Wij)$ is error gradient with reference to the weight Wij .

  Note : The Hoffs Delta rule and Back-propagation learning rule are the examples of Gradient descent learning.

- **Competitive Learning**

  - In this method, those neurons which respond strongly to the input stimuli have their weights updated.
  - When an input pattern is presented, all neurons in the layer compete, and the winning neuron undergoes weight adjustment .
  - This strategy is called "winner-takes-all".

- **Stochastic Learning**
  - In this method the weights are adjusted in a probabilistic fashion.
  - Example : Simulated annealing which is a learning mechanism employed by Boltzmann and Cauchy machines.

5. Taxonomy Of Neural Network Systems

In the previous sections, the Neural Network Architectures and the Learning methods

have been discussed. Here the popular neural network systems are listed. The grouping of these systems in terms of architectures and the learning methods are presented in the next slide.

- **Neural Network Systems**
  - ADALINE (Adaptive Linear Neural Element)
  - ART (Adaptive Resonance Theory)
  - AM (Associative Memory)
  - BAM (Bidirectional Associative Memory)
  - Boltzmann machines
  - BSB (Brain-State-in-a-Box)
  - Cauchymachines
  - HopfieldNetwork
  - LVQ (Learning Vector Quantization)
  - Neoconition
  - Perceptron
  - RBF (Radial Basis Function)
  - RNN (Recurrent Neural Network)
  - SOFM (Self-organizing Feature Map)

- **Classification of Neural Network**

  A taxonomy of neural network systems based on Architectural types and the Learning methods is illustrated below.

		Learning Methods			
		Gradient descent	**Hebbian**	**Competitive**	**Stochastic**
	Single-layer feed-forward	**ADALINE, Hopfield, Percepton,**	**AM, Hopfield,**	**LVQ, SOFM**	**-**

Types of Architecture	Multi-layer feed- forward	CCM, MLFF, RBF	Neocognition		
	Recurrent Networks	RNN	BAM, BSB, Hopfield,	ART	Boltzmann and Cauchy machines

Table : Classification of Neural Network Systems with respect to learning methods and Architecture types

6. Single-Layer NN Systems

Here, a simple Perceptron Model and an ADALINE Network Model is presented.

Single layer Perceptron

Definition : An arrangement of one input layer of neurons feed forward to one output layer of neurons is known as Single Layer Perceptron.

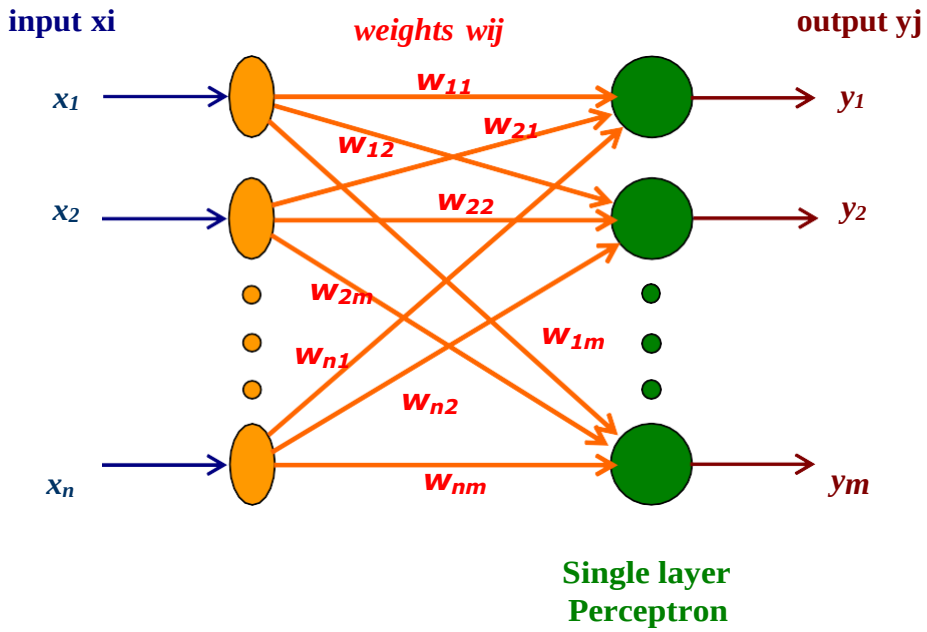


Fig. Simple Perceptron Model

$$y_j = f(net_j) = \begin{cases} 1 & \text{if } net_j \geq 0 \\ 0 & \text{if } net_j < 0 \end{cases} \quad \text{where } net_j = \sum_{i=1}^{n} x_i w_{ij}$$

- **Learning Algorithm : Training Perceptron**

The training of Perceptron is a supervised learning algorithm where weights are adjusted to minimize error when ever the output does not match the desired output.

- If the output is correct then no adjustment of weights is done.

  i.e. $W_{ij}^{K+1} = W_{ij}^{K}$

- If the output is **1** but should have been **0** then the weights are decreased on the active input link

  i.e. $W_{ij}^{K+1} = W_{ij}^{K} - \square . x_i$

- If the output is **0** but should have been **1** then the weights are increased on the active input link

  i.e. $W_{ij}^{K+1} = W_{ij}^{K} + \square . x_i$

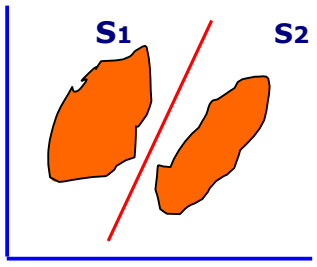
Where W_{ij}^{K+1} is the new adjusted weight, W_{ij}^{K} is the old weight

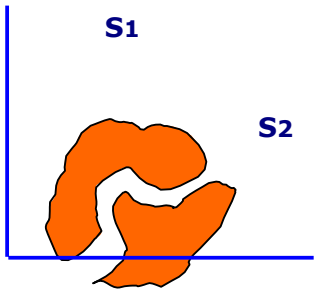
- **Perceptron and Linearly Separable Task**

Perceptron can not handle tasks which are not separable.

- Definition : Sets of points in 2-D space are linearly separable if the sets can be separated by a straight line.
- Generalizing, a set of points in n-dimensional space are linearly separable if there is a hyper plane of (n-1) dimensions separates the sets.

Example





(a) Linearly separable patterns **(b) Not Linearly separable patterns**

Note : Perceptron cannot find weights for classification problems that are not linearly separable.

- **XOR Problem :**

Exclusive OR operation

Input x1	Input x2	Output	
0	0	0	Even parity
1	1	0	
0	1	1	Odd parity
1	0	1	

XOR truth table

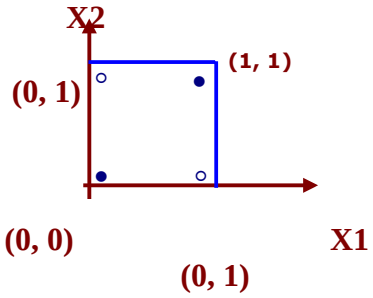


Fig. Output of XOR in X1 , x2 plane

Even parity is, even number of 1 bits in the input Odd parity is, odd number of 1 bits in the input

- There is no way to draw a single straight line so that the circles are on one side of the line and the dots on the other side.
- Perceptron is unable to find a line separating even parity input patterns from odd parity input patterns.

- **Perceptron Learning Algorithm**

The algorithm is illustrated step-by-step.

- **Step 1 :**

  Create a peceptron with **(n+1)** input neurons **x0 , x1 , ..., . xn ,** where **x0 = 1** is the bias input. Let **O** be the output neuron.

- **Step 2 :**

  Initialize weight **W = (w0 , w1 , ... , . wn)** to random weights.

- **Step 3 :**

  Iterate through the input patterns ***Xj*** of the training set using the weight set; ie compute the weighted sum of inputs **net j** = $\sum_{i=1}^{n} x_i w_i$ for each input pattern **j .**

- **Step 4 :**

  Compute the output **y j** using the step function

$$y_j = f(net_j) = \begin{cases} 1 & \text{if } net_j \geq 0 \\ 0 & \text{if } net_j < 0 \end{cases} \quad \text{where} \quad net_j = \sum_{i=1}^{n} x_i w_{ij}$$

- **Step 5 :**

  Compare the computed output **yj** with the target output **yj** for each input pattern ***j*** .

  If all the input patterns have been classified correctly, then output (read) the weights and exit.

- **Step 6 :**

  Otherwise, update the weights as given below :

  If the computed outputs **yj** is **1** but should have been **0**,

  Then **wi = wi - □ xi ,** **i= 0, 1, 2, ... , n**

  If the computed outputs **yj** is **0** but should have been **1**,

  Then **wi = wi + □ xi ,** **i= 0, 1, 2, ..., n**

  where □ is the learning parameter and is constant.

- **Step 7 :**

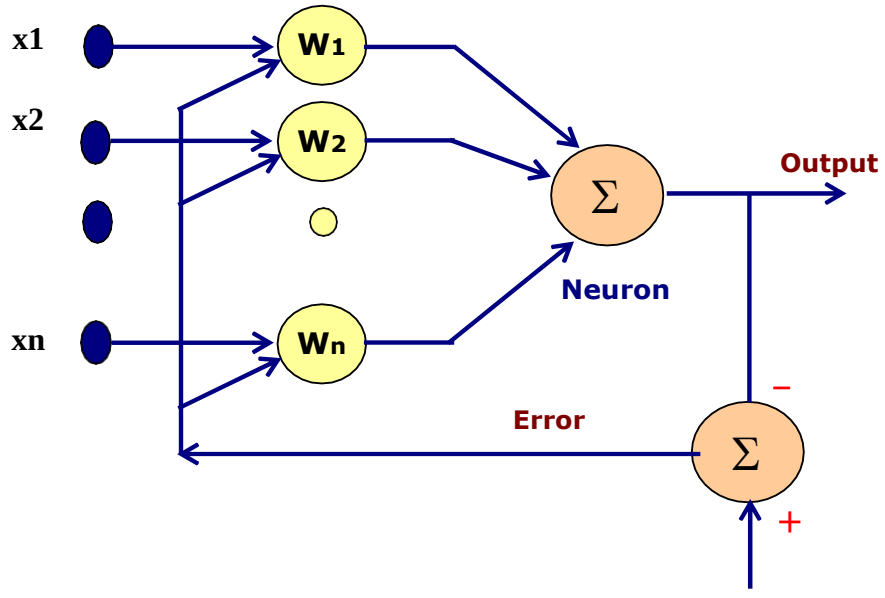
  goto step 3

- **END**

ADAptive LINear Element (ADALINE)

An ADALINE consists of a single neuron of the McCulloch-Pitts type, where its weights are determined by the normalized least mean square (LMS) training law. The LMS learning rule is also referred to as delta rule. It is a well-established supervised training method that has been used over a wide range of diverse applications.

- **Architecture of a simple ADALINE**



Desired Output

The basic structure of an ADALINE is similar to a neuron with a linear activation function and a feedback loop. During the training phase of ADALINE, the input vector as well as the desired output are presented to the network.

[The complete training mechanism has been explained in the next slide.]

- **ADALINE Training Mechanism**

  (Ref. Fig. in the previous slide - Architecture of a simpleADALINE)

  - The basic structure of an ADALINE is similar to a linear neuron with an extra feedback loop.
  - During the training phase of ADALINE, the input vector **X = [x1 , x2 , . . ., xn]^T** as well as desired output are presented to the network.
  - The weights are adaptively adjusted based on delta rule.
  - After the ADALINE is trained, an input vector presented to the network with fixed weights will result in a scalar output.
  - Thus, the network performs an **n** dimensional mapping to a scalar value.
  - The activation function is not used during the training phase. Once the weights are properly adjusted, the response of the trained unit can be tested by applying various inputs, which are not in the training set. If the network produces consistent responses to a high degree with the test inputs, it is said that the network could generalize. The process of training and generalization are two important attributes of this network.

Usage of ADLINE :

In practice, an ADALINE is used to

- Make binary decisions; the output is sent through a binary threshold.
- Realizations of logic gates such as AND, NOT and OR .
- Realize only those logic functions that are linearly separable.

Applications of Neural Network

Neural Network Applications can be grouped in following categories:

- **Clustering:**

  A clustering algorithm explores the similarity between patterns and places similar patterns in a cluster. Best known applications include data compression and data mining.

- **Classification/Pattern recognition:**

  The task of pattern recognition is to assign an input pattern (like handwritten symbol) to one of many classes. This category includes algorithmic implementations such as associative memory.

- **Function approximation :**

  The tasks of function approximation is to find an estimate of the unknown function subject to noise. Various engineering and scientific disciplines require function approximation.

- **Prediction Systems:**

  The task is to forecast some future values of a time-sequenced data. Prediction has a significant impact on decision support systems. Prediction differs from function approximation by considering time factor. System may be dynamic and may produce different results for the same input data based on system state (time).

Back Propagation Network

Soft Computing

Back-Propagation Network, topics : Background, what is back-prop network ? learning AND function, simple learning machines - Error measure , Perceptron learning rule, Hidden Layer, XOR problem. Back-Propagation Learning : learning by example, multi-layer feed-forward back-propagation network, computation in input, hidden and output layers, error calculation. Back-propagation algorithm for training network - basic loop structure, step-by-step procedure, numerical example.

1. **Back-Propagation Learning - learning by example**

   Multi-layer Feed-forward Back-propagation network; Computation of Input, Hidden and Output layers ; Calculation of Error.

2. **Back-Propagation Algorithm**

   Algorithm for training Network - Basic loop structure, Step-by-step procedure; Example: Training Back-prop network, Numerical example.

Back-Propagation Network

What is BPN ?

- A single-layer neural network has many restrictions. This network can accomplish very limited classes of tasks.

  Minsky and Papert (1969) showed that a two layer feed-forward network can overcome many restrictions, but they did not present a solution to the problem as "how to adjust the weights from input to hidden layer" ?

- An answer to this question was presented by Rumelhart, Hinton and Williams in 1986. The central idea behind this solution is that the errors for the units of the hidden layer are determined by back-propagating the errors of the units of the output layer.

  This method is often called the Back-propagation learning rule.

  Back-propagation can also be considered as a generalization of the delta rule for non-linear activation functions and multi-layer networks.

- Back-propagation is a systematic method of training multi-layer artificial neural networks.

1. Back-Propagation Network – Background

Real world is faced with a situations where data is incomplete or noisy. To make reasonable predictions about what is missing from the information available is a difficult task when there is no a good theory available that may to help reconstruct the missing data. It is in such situations the Back-propagation (Back-Prop) networks may provide some answers.

- A BackProp network consists of at least three layers of units :
  - an **input layer**,
  - at least one intermediate **hidden layer**, and
  - an **output layer**.

- Typically, units are connected in a feed-forward fashion with input units fully connected to units in the hidden layer and hidden units fully connected to units in the output layer.

- When a BackProp network is cycled, an input pattern is propagated forward to the output units through the intervening input-to-hidden and hidden-to-output weights.

- The output of a BackProp network is interpreted as a classification decision.

- With BackProp networks, learning occurs during a training phase. The steps followed during learning are :

  – each input pattern in a training set is applied to the input units and then propagated forward.

  – the pattern of activation arriving at the output layer is compared with the correct (associated) output pattern to calculate an error signal.

  – the error signal for each such target output pattern is then back-propagated from the outputs to the inputs in order to appropriately adjust the weights in each layer of the network.

  – after a BackProp network has learned the correct classification for a set of inputs, it can be tested on a second set of inputs to see how well it classifies

untrained patterns.

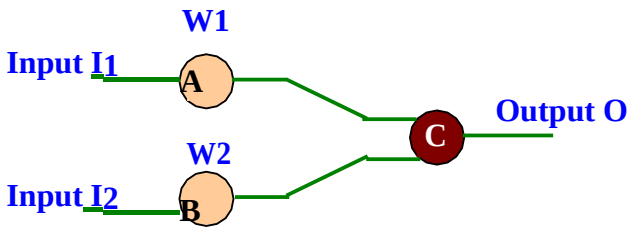
- An important consideration in applying BackProp learning is how well the network generalizes.

Learning :

AND function

Implementation of AND function in the neural network.

AND		
X1	X2	Y
0	0	0
0	1	0
1	0	0
1	1	1



AND function implementation

- there are 4 inequalities in the AND function and they must be satisfied.

  $w_1 0 + w_2 0 < \theta$, $w_1 0 + w_2 1 < \theta$,

  $w_1 1 + w_2 0 < \theta$, $w_1 1 + w_2 1 > \theta$

- one possible solution :

  if both weights are set to 1 and the threshold is set to 1.5, then

  (1)(0) + (1)(0) < 1.5 assign 0 , **(1)(0) + (1)(1) < 1.5 assign 0**

  (1)(1) + (1)(0) < 1.5 assign 0 , **(1)(1) + (1)(1) > 1.5 assign 1**

Although it is straightforward to explicitly calculate a solution to the AND function problem, but the question is "how the network can learn such a solution". That is, given random values for the weights can we define an incremental procedure which will cover a set of weights which implements AND function.

- **Example 1**

  AND Problem

  Consider a simple neural network made up of two inputs connected to a single output unit.

AND		
X1	X2	Y
0	0	0
0	1	0
1	0	0
1	1	1

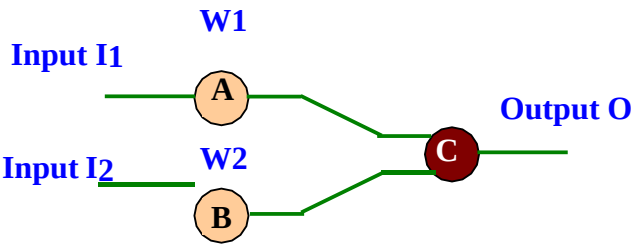


Fig A simple two-layer network applied to the AND problem

- the output of the network is determined by calculating a weighted sum of its two inputs and comparing this value with a threshold **θ**.
- if the net input (**net**) is greater than the threshold, then the output is **1**, else it is **0**.
- mathematically, the computation performed by the output unit is

net = w1 I1 + w2 I2 if net > θ then O = 1, otherwise O = 0.

- **Example 2**

Marital status and occupation

In the above example 1

- the input characteristics may be : **marital Status (single or married)** and their **occupation (pusher or bookie)**.
- this information is presented to the network as a 2-D binary input vector where 1st element indicates **marital status (single = 0, married = 1)** and 2nd element indicates **occupation (pusher = 0, bookie = 1).**
- the output, comprise **"class 0"** and **"class 1".**
- by applying the **AND operator** to the inputs, we classify an individual as a member of the **"class 0"** only if they are both married and a bookie; that is the output is **1** only when both of the inputs are **1**.

Simple Learning Machines

Rosenblatt (late 1950's) proposed learning networks called **Perceptron**. The task was to discover a set of connection weights which correctly classified a set of binary input vectors. The basic architecture of the perceptron is similar to the simple AND network in the previous example.

A perceptron consists of a set of input units and a single output unit. As in the AND network, the output of the perceptron is calculated by comparing the net input **net** = $\sum_{i=1}^{n}$ **wi Ii** and a threshold **θ.**

If the net input is greater than the threshold **θ** , then the output unit is turned **on** , otherwise it is turned **off**.

To address the learning question, Rosenblatt solved two problems.

- first, defined a cost function which measurederror.
- second, defined a procedure or a rule which reduced that error by appropriately adjusting each of the weights in the network.

However, the procedure (or **learning rule**) required to assesses the relative contribution of each weight to the total error.

The learning rule that Roseblatt developed, is based on determining the difference between the actual output of the network with the target output (**0** or **1**), called "**error measure**" which is explained in the next slide.

- **Error Measure** (learning rule)

Mentioned in the previous slide, the error measure is the difference between actual output of the network with the target output (**0** or **1**).

- If the input vector is correctly classified (i.e., zero error), then the weights are left unchanged, and the next input vector is presented.
- If the input vector is incorrectly classified (i.e., not zero error), then there are two cases to consider:

  Case 1 : If the output unit is **1** but need to be **0** then

  ◊ the threshold is incremented by **1** (to make it less likely that the output unit would be turned on if the same input vector was presented again).

  ◊ If the input **Ii** is **0**, then the corresponding weight W_i is left unchanged.

  ◊ If the input **Ii** is **1**, then the corresponding weight W_i is decreased by **1**.

  Case 2 : If output unit is **0** but need to be **1** then the opposite changes are made.

The perceptron learning rules are govern by two equations,

- one that defines the change in the threshold and
- the other that defines change in the weights, The

change in the threshold is givenby

$$\Delta\theta = -(t_p - o_p) = -d_p$$

where **p** specifies the presented input pattern,

o_p actual output of the input pattern I_{pi}

t_p specifies the correct classification of the input pattern ie target,

d_p is the difference between the target and actual outputs.

The **change in the weights** are given by

$$\Delta w_i = (t_p - o_p) I_{pi} = -d_p I_{pi}$$

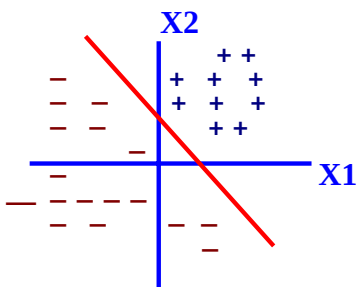
Hidden Layer

Back-propagation is simply a way to determine the error values in hidden layers. This needs be done in order to update the weights.

The best example to explain where back-propagation can be used is the XOR problem.

Consider a simple graph shown below.

- all points on the right side of the line are +ve, therefore the output of the neuron should be **+ve**.
- all points on the left side of the line are **–ve**, therefore the output of the neuron should be **–ve**.



With this graph, one can make a simple table of inputs and outputs as shown below.

AND

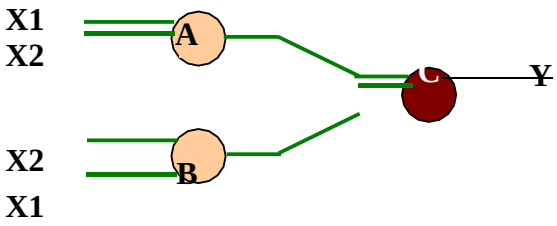
X1	*X2*	*Y*
1	1	1
1	0	0
0	1	0
0	0	0

Training a network to operate as an AND switch can be done easily through only one neuron *(see previous slides)*

But a XOR problem can't be solved using only one neuron.

If we want to train an XOR, we need 3 neurons, fully-connected in a feed-forward network as shown below.

XOR		
X1	*X2*	*Y*
1	1	0
1	0	1
0	1	1
0	0	0



2. Back Propagation Network

Learning By Example

Consider the Multi-layer feed-forward back-propagation network below. The subscripts **I, H, O** denotes input, hidden and output neurons.

The weight of the arc between **i th** input neuron to **j th** hidden layer is **Vij** .

The weight of the arc between **i th** hidden neuron to **j th** out layer is **Wij**

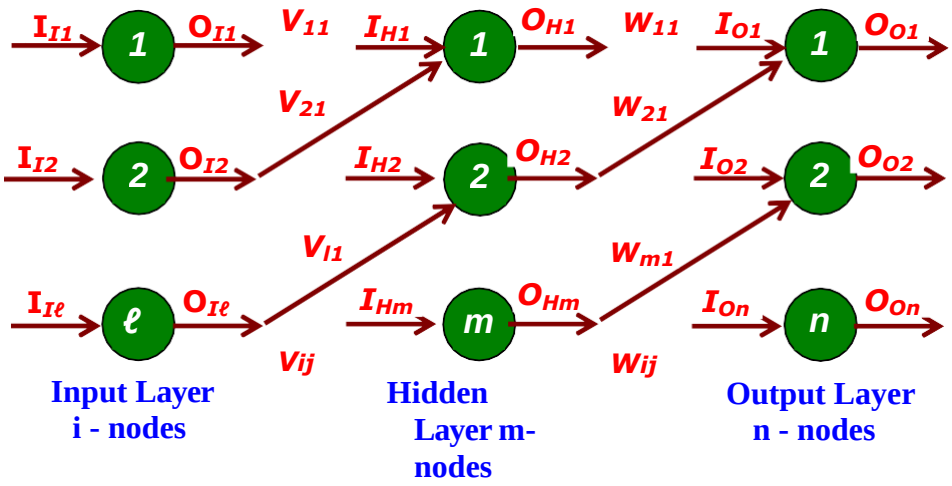


Fig Multi-layer feed-forward back-propagation network

The table below indicates an 'nset' of input and out put data. It shows **ℓ** inputs and the corresponding **n** outputdata.

Table : 'nset' of input and output data

No	Input				Ouput			
	I1	I2	. . . .	Iℓ	O1	O2	. . . .	On
1	0.3	0.4	. . . .	0.8	0.1	0.56	.....	0.82
2								
:								
nset								

In this section, over a three layer network the computation in the input, hidden and output layers are explained while the step-by-step implementation of the BPN algorithm by solving an example is illustrated in the next section.

Computation of Input, Hidden and Output Layers

(Ref.Previous slide, Fig. Multi-layer feed-forward back-propagation network)

- **Input Layer Computation**

Consider linear activation function.

If the output of the input layer is the input of the input layer and the transfer function is **1**, then

$$\{ O \}_I = \{ I \}_I$$

ℓ x 1 ℓ x 1 (denotes matrix row, column size)

The hidden neurons are connected by synapses to the input neurons.

- Let V_{ij} be the weight of the arc between i^{th} input neuron to j^{th} hidden layer.
- The input to the hidden neuron is the weighted sum of the outputs of the input neurons. Thus the equation

$$IHp = V1p\ OI1 + V2p\ OI2 + \ldots + V1p\ OI\ell \text{ where } (p = 1, 2, 3 \ldots, m)$$

denotes weight matrix or connectivity matrix between input neurons and a hidden neurons as **[V]**.

we can get an input to the hidden neuron as **ℓ x m**

$$\{ I \}_H = [V]^T \{ O \}_I$$

m x 1 m x ℓ ℓ x 1 (denotes matrix row, column size)

Hidden Layer Computation

Shown below the p^{th} neuron of the hidden layer. It has input from the output of the input neurons layers. If we consider transfer function as

sigmoidal function then the output of the p^{th} hidden neuron is given by

$$OHp = \frac{1}{(1 + e^{-\square\ (I_{HP} - \theta_{HP})})}$$

where **OHp** is the output of the p^{th} hidden neuron, **IHp** is the input of the p^{th} hidden neuron, and **θHP** is the threshold of the p^{th} neuron;

Note : a non zero threshold neuron, is computationally equivalent to an input that is always held at **-1** and the non-zero threshold becomes the connecting weight value as shown in Fig. below.

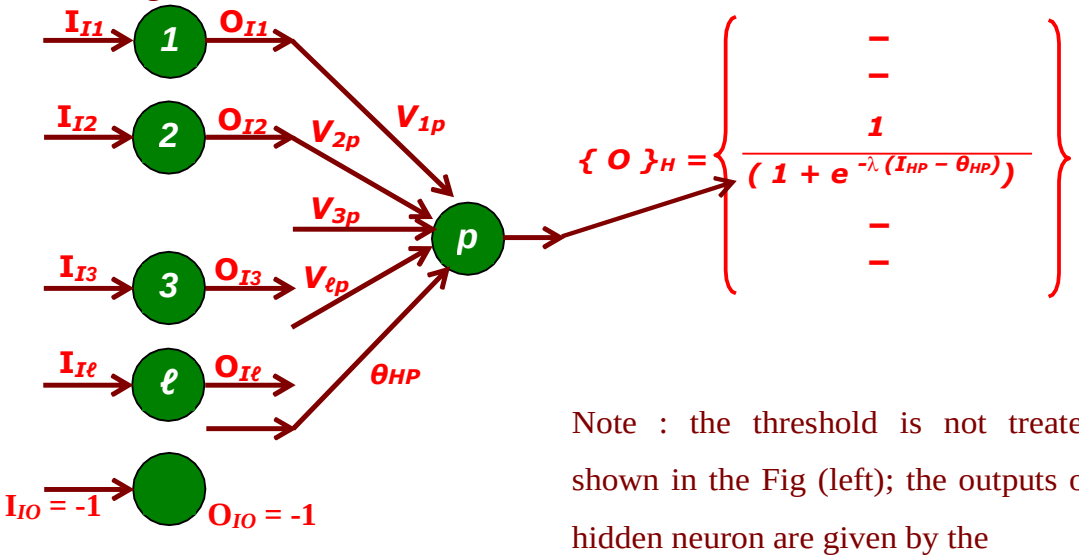


Fig. Example of Treating threshold in hidden layer

Note : the threshold is not treated as shown in the Fig (left); the outputs of the hidden neuron are given by the above equation.

Treating each component of the input of the hidden neuron separately, we get the outputs of the hidden neuron as given by above equation .

The input to the output neuron is the weighted sum of the outputs of the hidden neurons. Accordingly, **Ioq** the input to the **q^{th}** output neuron is given by the equation

$$I_{oq} = W_{1q} O_{H1} + W_{2q} O_{H2} + \ldots + W_{mq} O_{Hm} \text{ , where } (q = 1, 2, 3 \ldots, n)$$

It denotes weight matrix or connectivity matrix between hidden neurons and output neurons as **[W]**, we can get input to output neuron as

$$\{ I \}_O = [W]^T \{ O \}_H$$

n x 1 n x m m x 1 (denotes matrix row, column size)

- **Output Layer Computation**

Shown below the q^{th} neuron of the output layer. It has input from the output of the hidden neurons layers.

If we consider transfer function as sigmoidal function then the output of the q^{th} output neuron is given by

$$OOq = \frac{1}{(1 + e^{-\lambda (I_{Oq} - \theta_{Oq})})}$$

where OOq is the output of the q^{th} output neuron,

IOq is the input to the q^{th} output neuron, and

θOq is the threshold of the q^{th} neuron;

Note : A non zero threshold neuron, is computationally equivalent to an input that is always held at **-1** and the non-zero threshold becomes the connecting weight value as shown in Fig. below.

Note : Here again the threshold may be tackled by considering extra O^{th} neuron in the hidden layer with output of **-1** and the threshold value θOq becomes the connecting weight value as shown in Fig. below.

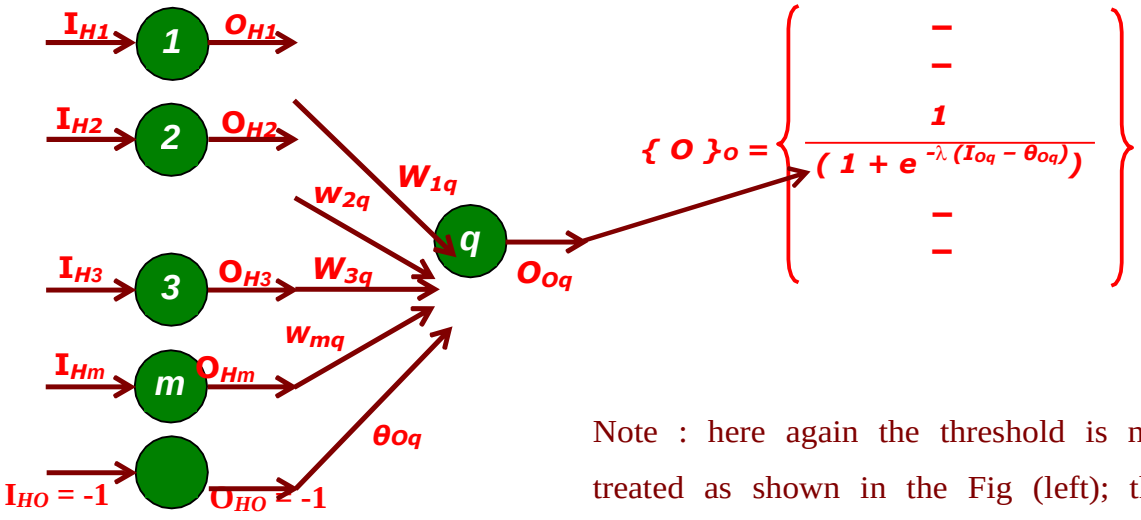


Fig. Example of Treating threshold in output layer

Note : here again the threshold is not treated as shown in the Fig (left); the Outputs of the output neurons given by the above equation.

Calculation of Error

(refer the earlier slides - Fig. "Multi-layer feed-forward back-propagation network" and a table indicating an 'nset' of input and out put data for the purpose of training*)*

Consider any **r th** output neuron. For the target out value **T**, mentioned in the table- 'nset' of input and output data" for the purpose of training, calculate output **O** .

The error norm in output for the **r th** output neuron is

$$E^1r = (1/2)\ e^2r = (1/2)\ (T - O)^2$$

where E^1r is **1/2** of the second norm of the error **er** in the **r th** neuron for the given training pattern.

e^2r is the square of the error, considered to make it independent of sign **+ve** or **–ve** , ie consider only the absolute value.

The Euclidean norm of error E^1 for the first training pattern is given by

$$E^1 = (1/2) \sum_{r=1}^{n} (Tor - Oor)^2$$

This error function is for one training pattern. If we use the same technique for all the training pattern, we get

$$E\ (V, W) = \sum_{r=1}^{nset} E^j\ (V, W, I)$$

where **E** is error function depends on **m (1 + n)** weights of **[W]** and **[V]**.

All that is stated is an **optimization** problem solving, where the objective or cost function is usually defined to be maximized or minimized with respect to a set of parameters. In this case, the network parameters that optimize the error function **E** over the '**nset**' of pattern sets **[I ^{nset} , t ^{nset}]** are synaptic weight values **[V]** and **[W]** whose sizes are

[V] and **[W]**
ℓ x m m x n

16 Back-Propagation Algorithm

The benefits of hidden layer neurons have been explained. The hidden layer allows ANN to develop its own internal representation of input-output mapping. The complex internal representation capability allows the hierarchical network to learn any mapping and not just the linearly separable ones.

The step-by-step algorithm for the training of Back-propagation network is presented in next few slides. The network is the same , illustrated before, has a three layer. The input layer is with **ℓ** nodes, the hidden layer with **m** nodes and the output layer with **n** nodes. An example for training a BPN with five training set have been shown for better understanding.

Algorithm for Training Network

The basic algorithm loop structure, and the step by step procedure of Back-propagation algorithm are illustrated in next few slides.

- **Basic algorithm loop structure**

Initialize the weights Repeat
For each training pattern
"Train on that pattern"
End
Until the error is acceptably low.

- **Back-Propagation Algorithm -** Step-by-step procedure

  - **Step 1 :**

    Normalize the I/P and O/P with respect to their maximum values.

    For each training pair, assume that in normalized form there are

    ℓ inputs given by **{ I }$_I$** and

    ℓ x 1

    n outputs given by **{ O}$_O$**

    n x 1

  - **Step 2 :**

    Assume that the number of neurons in the hidden layers lie between **1 < m < 21**

- **Step 3 :**

Let **[V]** represents the weights of synapses connecting input neuron and hidden neuron

Let **[W]** represents the weights of synapses connecting hidden neuron and output neuron

Initialize the weights to small random values usually from **-1 to +1;**

[V] ^{0} = [random weights] [W

] ^{0} = [random weights] [□ V] ^{0}

= [□ W] ^{0} = [0]

For general problems □ can be assumed as **1** and threshold value as **0**.

- **Step 4 :**

  For training data, we need to present one set of inputs and outputs. Present the pattern as inputs to the input layer $\{ I \}_I$.

  then by using linear activation function, the output of the input layer may be evaluated as

  $$\{ O \}_I = \{ I \}_I$$
  $$\ell \times 1 \qquad \ell \times 1$$

- **Step 5 :**

  Compute the inputs to the hidden layers by multiplying corresponding weights of synapses as

  $$\{ I \}_H = [V]^T \{ O \}_I$$
  $$m \times 1 \qquad m \times \ell \quad \ell \times 1$$

- **Step 6 :**

  Let the hidden layer units, evaluate the output using the sigmoidal function as

$$\{ O \}_H = \left\{ \begin{array}{c} - \\ - \\ \dfrac{1}{(1 + e^{-(I_{Hi})})} \\ - \\ - \end{array} \right\}$$

$$m \times 1$$

- **Step 9 :**

  Compute the inputs to the output layers by multiplying corresponding weights of synapses as

$$\{ I \}_O = [W]^T \{ O \}_H$$

  n x 1 n x m m x 1

- **Step 8 :**

  Let the output layer units, evaluate the output using sigmoidal function as

$$\{ O \}_O = \left\{ \frac{1}{(1 + e^{- (IOj)})} \right\}$$

  Note : This output is the network output

  Calculate the error using the difference between the network output and the desired output as for the **j** th training set as

$$E^P = \sqrt{\frac{\sum (Tj - Ooj)^2}{n}}$$

- **Step 10 :**

  Find a term **{ d }** as

$$\{ d \} = \begin{Bmatrix} \\ (Tk - OOk)\ OOk\ (1 - OOk) \\ \\ \end{Bmatrix}$$

  n x 1

- **Step 11 :**

Find **[Y]** matrix as

$$\underset{m \times n}{[Y]} = \underset{m \times 1}{\{O\}_H} \; \square \; \underset{1 \times n}{d} \; \square$$

- **Step 12 :**

Find $$\underset{m \times n}{[\square W]^{t+1}} = \square \underset{m \times n}{[\square W]^{t}} + \square \underset{m \times n}{[Y]}$$

- **Step 13 :**

Find $$\underset{m \times 1}{\{e\}} = \underset{m \times n}{[W]} \underset{n \times 1}{\{d\}}$$

$$\{d^*\} = \begin{Bmatrix} - \\ - \\ (O_{Hi})\,(1 - O_{Hi}) \\ e_i \\ - \\ - \end{Bmatrix}$$

$m \times 1 \quad m \times 1$

Find **[X]** matrix as

$$\underset{1 \times m}{[X]} = \underset{\ell \times 1}{\{O\}_I} \; \square \; \underset{1 \times m}{d^*} \; \square = \underset{\ell \times 1}{\{I\}_I} \; \square \; \underset{1 \times m}{d^*} \; \square$$

- **Step 14 :**

  Find $[\Delta V]^{t+1}_{1 \times m} = \alpha [\Delta V]^{t}_{1 \times m} + \alpha [X]_{1 \times m}$

- **Step 15 :**

  Find $[V]^{t+1} = [V]^{t} + [\Delta V]^{t+1}$

  $[W]^{t+1} = [W]^{t} + [\Delta W]^{t+1}$

- **Step 16 :**

  Find error rate as

  $$\text{error rate} = \frac{\Delta}{Ep}$$

  $$nset$$

- **Step 17 :**

  Repeat steps 4 to 16 until the convergence in the error rate is less than the tolerance value

- **End of Algorithm**

  Note : The implementation of this algorithm, step-by-step 1 to 17, assuming one example for training BackProp Network is illustrated in the next section.

Example : **Training Back-Prop Network**

- **Problem :**

Consider a typical problem where there are 5 training sets.

Table : Training sets

S. No.	Input		Output
	I_1	I_2	O
1	0.4	-0.7	0.1
2	0.3	-0.5	0.05
3	0.6	0.1	0.3
4	0.2	0.4	0.25
5	0.1	-0.2	0.12

In this problem,

- there are two inputs and one output.
- the values lie between -1 and +1 i.e., no need to normalize the values.
- assume two neurons in the hiddenlayers.
- the NN architecture is shown in the Fig. below.

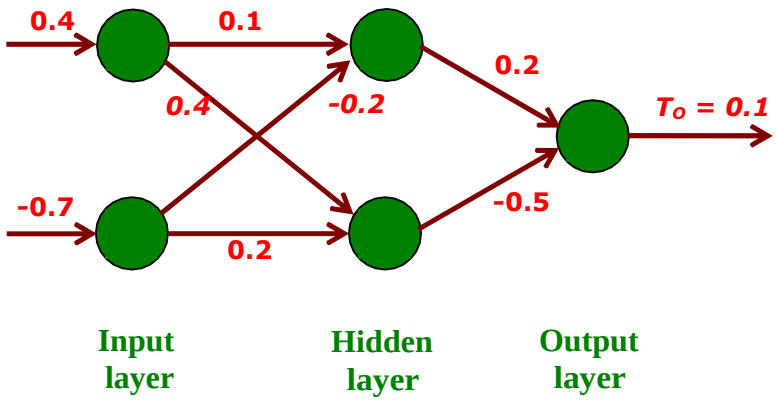


Fig. Multi layer feed forward neural network (MFNN) architecture with data of the first training set

The solution to problem are stated step-by-step in the subsequent slides.

- **Step 1 :** Input the first training set data *(ref eq. of step 1)*

$$\{O\}_I = \underset{\ell \times 1}{} \qquad \{I\}_I = \underset{\ell \times 1}{} \qquad \begin{Bmatrix} 0.4 \\ -0.7 \end{Bmatrix}_{2 \times 1}$$

from training set s.no 1

- **Step 2 :** Initialize the weights as *(ref eq. of step 3 & Fig)*

$$[V]^0 = \underset{2x2}{\begin{Bmatrix} 0.1 & 0.4 \\ -0.2 & 0.2 \end{Bmatrix}} \quad ; \quad [W]^0 = \underset{2 \times 1}{\begin{Bmatrix} 0.2 \\ -0.5 \end{Bmatrix}}$$

from fig initialization *from fig initialization*

- **Step 3 :** Find $\{I\}_H = [V]^T \{O\}_I$ as *(ref eq. of step 5)*

$$\{I\}_H = \begin{Bmatrix} 0.1 & -0.2 \\ -0.4 & 0.2 \end{Bmatrix} \begin{Bmatrix} 0.4 \\ -0.7 \end{Bmatrix} = \begin{Bmatrix} 0.18 \\ 0.02 \end{Bmatrix}$$

- **Step 4 :**

(ref eq. of step 6)

Values from step 1 & 2

$$\left\{ \begin{array}{c} \dfrac{1}{(1 + e^{-(0.18)})} \\ \\ \dfrac{1}{(1 + e^{-(0.02)})} \end{array} \right\} \quad \left\{ \begin{array}{c} 0.5448 \\ \\ 0.505 \end{array} \right\}$$

$$\{ O \}_H =$$

- **Step 5 :** *(ref eq. of step 7)*

Values from step 3 values

$$\{ I \}_O = [W]^T \{ O \}_H = (0.2 \;\; -0.5) \begin{Bmatrix} 0.5448 \\ 0.505 \end{Bmatrix} = -0.14354$$

Values from step 2 , from step 4

- **Step 6 :** *(ref eq. of step 8)*

$$\{ O \}_O = \left\{ \frac{1}{(1 + e^{-(0.14354)})} \right\} = 0.4642$$

Values from step 5

- **Step 7 :** *(ref eq. of step 9)*

$$\text{Error} = (T_O - O_{O1})^2 = (0.1 - 0.4642)^2 = 0.13264$$

table first training set o/p — *from step 6*

- **Step 8 :** *(ref eq. of step 10)*

$$d = (T_O - O_{O1})\,(O_{O1})\,(1 - O_{O1})$$

$$= (0.1 - 0.4642)\,(0.4642)\,(0.5358) = -0.09058$$

Training o/p — *all from step 6*

(ref eq. of step 11)

$$[Y] = \{O\}_H\,(d) = \begin{Bmatrix} 0.5448 \\ 0.505 \end{Bmatrix} (-0.09058) = \begin{Bmatrix} -0.0493 \\ -0.0457 \end{Bmatrix}$$

from values at step 4 — *from values at step 8 above*

- **Step 9 :** *(ref eq. of step 12)*

$$[\Delta W]^1 = \alpha\,[\Delta W]^0 + \eta\,[Y] \qquad \text{assume } \alpha = 0.6$$

$$= \begin{Bmatrix} -0.02958 \\ -0.02742 \end{Bmatrix}$$

from values at step 2 & step 8 above

- **Step 10 :** *(ref eq. of step 13)*

$$\{e\} = [W]\,\{d\} = \begin{Bmatrix} 0.2 \\ -0.5 \end{Bmatrix} (-0.09058) = \begin{Bmatrix} -0.018116 \\ -0.04529 \end{Bmatrix}$$

from values at step 2 — *from values at step 8 above*

- **Step 11 :** *(ref eq. of step 13)*

$$\{ d^* \} = \begin{Bmatrix} (-0.018116)\,(0.5448)\,(1-0.5448) \\ (0.04529)\,(0.505)\,(1-0.505) \end{Bmatrix} = \begin{Bmatrix} -0.00449 \\ -0.01132 \end{Bmatrix}$$

from values at step 10 *at step 4* *at step 8*

- **Step 12 :** *(ref eq. of step 13)*

$$[X] = \{ O \}_I (d^*) = \begin{Bmatrix} 0.4 \\ -0.7 \end{Bmatrix} (-0.00449 \quad 0.01132)$$

from values at step 1 *from values at step 11 above*

$$= \begin{Bmatrix} -0.001796 & 0.004528 \\ 0.003143 & -0.007924 \end{Bmatrix}$$

- **Step 13 :** *(ref eq. of step 14)*

$$[\Delta V]^1 = \alpha [\Delta V]^0 + \eta [X] = \begin{Bmatrix} -0.001077 & 0.002716 \\ 0.001885 & -0.004754 \end{Bmatrix}$$

from values at step 2 & step 8 above

■ **Step 14 :** *(ref eq. of step 15)*

$$[V]^1 = \begin{Bmatrix} 0.1 & 0.4 \\ -0.2 & 0.2 \end{Bmatrix} + \begin{Bmatrix} -0.001077 & 0.002716 \\ 0.001885 & -0.004754 \end{Bmatrix}$$

from values at step 2 *from values at step 13*

$$= \begin{Bmatrix} -0.0989 & 0.04027 \\ 0.1981 & -0.19524 \end{Bmatrix}$$

$$[W]^1 = \begin{Bmatrix} 0.2 \\ -0.5 \end{Bmatrix} + \begin{Bmatrix} -0.02958 \\ -0.02742 \end{Bmatrix} = \begin{Bmatrix} 0.17042 \\ -0.52742 \end{Bmatrix}$$

- **Step 14 :** *(ref eq. of step 15)*

  from values at step 2, *from values at step 9*

- **Step 15 :**

  With the updated weights **[V]** and **[W]** , error is calculated again and next training set is taken and the error will then get adjusted.

- **Step 16 :**

  Iterations are carried out till we get the error less than the tolerance.

- **Step 17 :**

  Once the weights are adjusted the network is ready for inferencing new objects .

Fuzzy Set Theory

Soft Computing

Introduction to fuzzy set, topics : classical set theory, fuzzy set theory, crisp and non-crisp Sets representation, capturing uncertainty, examples. Fuzzy membership and graphic interpretation of fuzzy sets - small, prime numbers, universal, finite, infinite, empty space; Fuzzy Operations - inclusion, comparability, equality, complement, union, intersection, difference; Fuzzy properties related to union, intersection, distributivity, law of excluded middle, law of contradiction, and cartesian product. Fuzzy relations : definition, examples, forming fuzzy relations, projections of fuzzy relations, max-min and min-max compositions.

Fuzzy Set Theory

Soft Computing

Topics

1. **Introduction to fuzzy Set**

   What is Fuzzy set? Classical set theory; Fuzzy set theory; Crisp and Non-crisp Sets : Representation; Capturing uncertainty, Examples

2. **Fuzzy set**

Fuzzy Membership; Graphic interpretation of fuzzy sets : small, prime numbers, universal, finite, infinite, empty space;

Fuzzy Operations : Inclusion, Comparability, Equality, Complement, Union, Intersection, Difference;

Fuzzy Properties : Related to union – Identity, Idempotence, Associativity, Commutativity ; Related to Intersection – Absorption, Identity, Idempotence, Commutativity, Associativity; Additional properties - Distributivity, Law of excluded middle, Law of contradiction; Cartesian product .

3. **Fuzzy Relations**

   Definition of Fuzzy Relation, examples;

   Forming Fuzzy Relations – Membership matrix, Graphical form; Projections of Fuzzy Relations – first, second and global; Max-Min and Min-Max compositions.

Fuzzy Set Theory

What is Fuzzy Set ?

- The word "fuzzy" means "vagueness". Fuzziness occurs when the boundary of a piece of information is not clear-cut.

- Fuzzy sets have been introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of set.

- Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition - an element either belongs or does not belong to the set.

  Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval [0,1].

- **Example:**

  Words like young, tall, good, or high are fuzzy.

  - There is no single quantitative value which defines the term young.
  - For some people, age 25 is young, and for others, age 35 is young.
  - The concept young has no clean boundary.
  - Age 1 is definitely young and age 100 is definitely not young;
  - Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

1. Introduction

In real world, there exists much fuzzy knowledge;

Knowledge that is vague, imprecise, uncertain, ambiguous, inexact, or probabilistic in nature.

Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Humans, can give satisfactory answers, which are probably true.

However, our systems are unable to answer many questions. The reason is, most systems are designed based upon classical set theory and two-valued logic which is unable to cope with unreliable and incomplete information and give expert opinions.

We want, our systems should also be able to cope with unreliable and incomplete information and give expert opinions. Fuzzy sets have been able provide solutions to many real world problems.

Fuzzy Set theory is an extension of classical set theory where elements have degrees of membership.

- **Classical Set Theory**

  A Set is any well defined collection of objects. An object in a set is called an element or member of that set.

  - Sets are defined by a simple statement describing whether a particular element having a certain property belongs to that particular set.
  - Classical set theory enumerates all its elements using

    $$A = \{ a_1 , a_2 , a_3 , a_4 , \ldots\ldots\ldots\ldots a_n \}$$

    If the elements a_i $(i = 1, 2, 3, \ldots n)$ of a set A are subset of universal set X, then set A can be represented for all elements $x \in X$ by its **characteristic function**

$$\mu_A(x) = \begin{cases} 1 \text{ if } x \in X \\ 0 \quad \text{otherwise} \end{cases}$$

- A set **A** is well described by a function called characteristic function.

  This function, defined on the universal space **X**, assumes :

  a value of **1** for those elements **x** that belong to set **A**, and

  a value of **0** for those elements **x** that do not belong to set **A**.

  The notations used to express these mathematically are

$$\left.\begin{array}{l} \mathbf{A : X \to [0, 1]} \\ \mathbf{A(x) = 1 , \ x \text{ is a member of } A} \\ \mathbf{A(x) = 0 , \ x \text{ is not a member of } A} \end{array}\right\} \quad \textbf{Eq.(1)}$$

  Alternatively, the set **A** can be represented for all elements **x** □ **X** by its characteristic function $\mu_A(x)$ defined as

$$\mu_A(x) = \begin{cases} \mathbf{1} & \mathbf{if \ \ x \in X} \\ \mathbf{0} & \mathbf{otherwise} \end{cases} \qquad \textbf{Eq.(2)}$$

- Thus in classical set theory $\mu_A(x)$ has only the values **0** ('false') and **1** ('true"). Such sets are called **crisp sets.**

- **Fuzzy Set Theory**

  Fuzzy set theory is an extension of classical set theory where elements have varying degrees of membership. A logic based on the two truth values, *True* and *False*, is sometimes inadequate when describing human reasoning. Fuzzy logic uses the whole interval between **0** (false) and **1** (true) to describe human reasoning.

  - A **Fuzzy Set** is any set that allows its members to have different degree of membership, called **membership function**, in the interval **[0 , 1]**.

  - The **degree of membership** or truth is not same as probability;
    - □ fuzzy truth is not likelihood of some event or condition.
    - □ fuzzy truth represents membership in vaguely defined sets;

  - **Fuzzy logic** is derived from fuzzy set theory dealing with reasoning that is approximate rather than precisely deduced from classical predicate logic.

  - Fuzzy logic is capable of handling inherently imprecise concepts.

- Fuzzy logic allows in linguistic form the set membership values to imprecise concepts like **"slightly", "quite" and "very".**

- Fuzzy set theory defines Fuzzy Operators on Fuzzy Sets.

- **Crisp and Non-Crisp Set**

  - As said before, in classical set theory, the **characteristic function** $\mu_A(x)$ of Eq.(2) has only values **0** ('false') and **1** ('true"). Such sets are **crisp sets.**

  - For Non-crisp sets the characteristic function $\mu_A(x)$ can be defined.
    - The characteristic function $\mu_A(x)$ of Eq. (2) for the crisp set is generalized for the Non-crisp sets.
    - This generalized characteristic function $\mu_A(x)$ of Eq.(2) is called **membership function**.

    Such Non-crisp sets are called **Fuzzy Sets**.

  - Crisp set theory is not capable of representing descriptions and classifications in many cases; In fact, Crisp set does not provide adequate representation for most cases.

  - The proposition of Fuzzy Sets are motivated by the need to capture and represent real world data with uncertainty due to imprecise measurement.

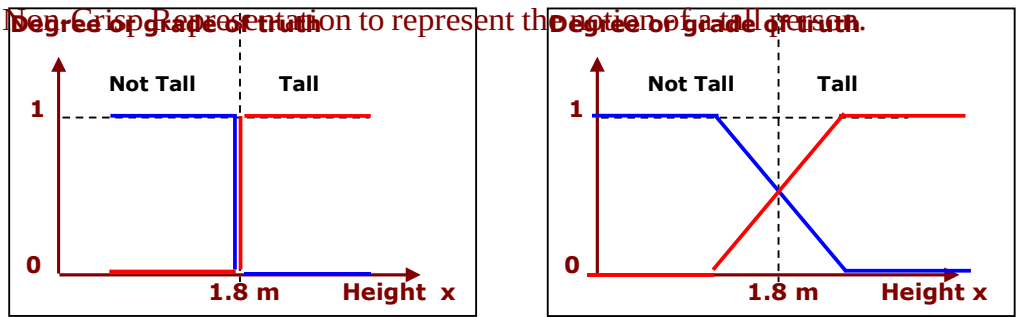
  - The uncertainties are also caused by vagueness in the language.

- **Representation of Crisp and Non-Crisp Set** Example :

  Classify students for a basketball team This example explains the grade of truth value.

  - **tall students** qualify and **not tall students** do not qualify
  - if students 1.8 m tall are to be qualified, then should we exclude a student who is $^1/_{10}$" less? or should we exclude a student who is 1" shorter?

  - Non-Crisp Representation to represent the notion of a tall person.



Crisp logic Non-crisp logic

Fig. 1 Set Representation – Degree or grade of truth

A student of height 1.79m would belong to both tall and not tall sets with a particular degree of membership.

As the height increases the membership grade within the tall set would increase whilst the membership grade within the not-tall set would decrease.

- **Capturing Uncertainty**

Instead of avoiding or ignoring uncertainty, Lotfi Zadeh introduced Fuzzy Set theory that captures uncertainty.

- A fuzzy set is described by a **membership function** $\mu_A(x)$ of **A**.
  This membership function associates to each element $x \in X$ a number as $\mu_A(x)$ in the closed unit interval **[0, 1]**.

  The number $\mu_A(x)$ represents the **degree of membership** of x in **A**.
- The notation used for membership function $\mu_A(x)$ of a fuzzy set **A** is

  $A : X \to [0, 1]$

- Each membership function maps elements of a given universal base set **X** , which is itself a crisp set, into real numbers in **[0, 1] .**

- Example

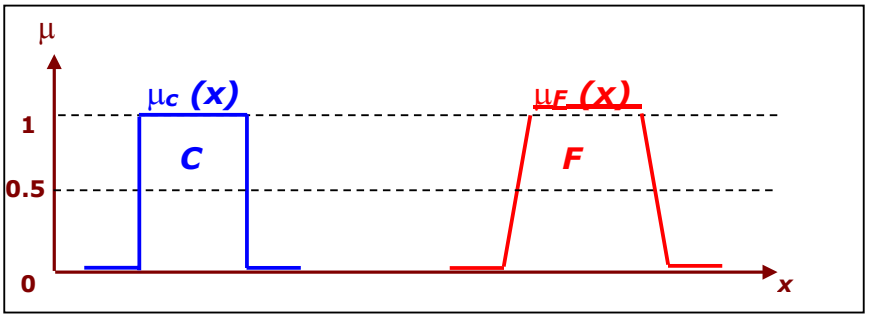


Fig. 2 Membership function of a Crisp set C and Fuzzy set F

- In the case of Crisp Sets the members of a set are :

  either out of the set, with membership of degree " **0** ", or in the set, with membership of degree " **1** ",

Therefore, **Crisp Sets ⊆ Fuzzy Sets**

In other words, Crisp Sets are Special cases of Fuzzy Sets.

- **Examples of Crisp and Non-Crisp Set**

  Example 1: Set of prime numbers (a crisp set)

  If we consider space **X** consisting of natural numbers **12**

  ie **X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}**

  Then, the set of prime numbers could be described as follows.

  PRIME = {x contained in X | x is a prime number} = {2, 3, 5, 6, 7, 11}

  Example 2: Set of SMALL (as non-crisp set)

  A Set **X** that consists of SMALL cannot be described;

  for example **1** is a member of SMALL and **12** is not a member of SMALL.

  Set **A**, as SMALL, has un-sharp boundaries, can be characterized by a function that assigns a real number from the closed interval from **0** to **1** to each element **x** in the set **X**.

A Fuzzy Set is any set that allows its members to have different degree of membership, called membership function, in the interval **[0 , 1]**.

- **Definition of Fuzzy set**

  A **fuzzy set A**, defined in the universal space **X**, is a function defined in **X** which assumes values in the range **[0, 1]**.

  A fuzzy set **A** is written as a set of pairs **{x, A(x)}** as

  A = {{x , A(x)} , **x in the set X**

  where **x** is an element of the universal space **X**, and

  A(x) is the value of the function **A** for this element.

  The value **A(x)** is the **membership grade** of the element **x** in a fuzzy set **A**.

  Example : Set **SMALL** in set **X** consisting of natural numbers **to 12**. **Assume:**

  SMALL(1) = 1, SMALL(2) = 1, SMALL(3) = 0.9, SMALL(4) = 0.6, SMALL(5) = 0.4, SMALL(6) = 0.3, SMALL(7) = 0.2, SMALL(8) = 0.1, SMALL(u) = 0 for u >= 9.

Then, following the notations described in the definition above :

Set SMALL = {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2},
{8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Note that a fuzzy set can be defined precisely by associating with each **x** , its grade of membership in**SMALL**.

- **Definition of Universal Space**

Originally the universal space for fuzzy sets in fuzzy logic was defined only on the integers. Now, the universal space for fuzzy sets and fuzzy relations is defined with three numbers.

The first two numbers specify the start and end of the universal space, and the third argument specifies the increment between elements. This gives the user more flexibility in choosing the universal space.

Example : The fuzzy set of numbers, defined in the universal space
X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as
SetOption [FuzzySet, UniversalSpace □ {1, 12, 1}]

Fuzzy Membership

A fuzzy set **A** defined in the universal space **X** is a function defined in **X** which assumes values in the range **[0, 1]**.

A fuzzy set **A** is written as a set of pairs **{x, A(x)}**.

A = {{x , A(x)}} , x in the set X

where **x** is an element of the universal space **X**, and

A(x) is the value of the function **A** for this element.

The value **A(x)** is the **degree of membership** of the element **x** in a fuzzy set **A**.

The Graphic Interpretation of fuzzy membership for the fuzzy sets : Small, Prime Numbers, Universal-space, Finite and Infinite UniversalSpace, and Empty are illustrated in the next few slides.

- **Graphic Interpretation of Fuzzy Sets SMALL**

The fuzzy set SMALL of small numbers, defined in the universal space

X = { xi } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

SetOption [FuzzySet, UniversalSpace □ {1, 12, 1}]

The Set **SMALL** in set **X** is :

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},
{7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetSmall** is represented as

SetSmall = FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.4},{6,0.3}, {7,0.2},
{8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

FuzzyPlot [SMALL, AxesLable □ {"X", "SMALL"}]

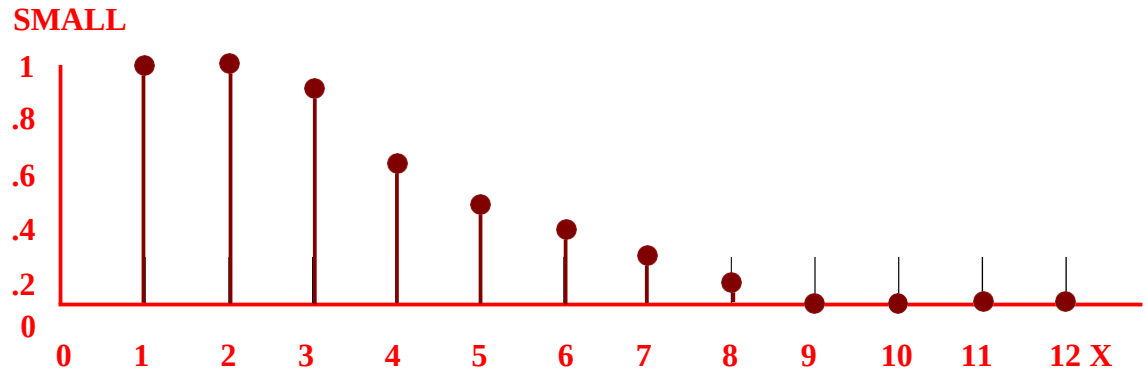


Fig Graphic Interpretation of Fuzzy Sets SMALL

- **Graphic Interpretation of Fuzzy Sets** **PRIME Numbers** The fuzzy set PRIME numbers, defined in the universal space **X = { xi } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace □ {1, 12, 1}]

The Set **PRIME** in set **X** is :

PRIME = FuzzySet {{1, 0}, {2, 1}, {3, 1}, {4, 0}, {5, 1}, {6, 0}, {7, 1}, {8, 0},
{9, 0}, {10, 0}, {11, 1}, {12, 0}}

Therefore **SetPrime** is represented as

SetPrime = FuzzySet [{{1,0},{2,1}, {3,1}, {4,0}, {5,1},{6,0}, {7,1},
{8, 0}, {9, 0}, {10, 0}, {11, 1}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

FuzzyPlot [PRIME, AxesLable □ {"X", "PRIME"}]

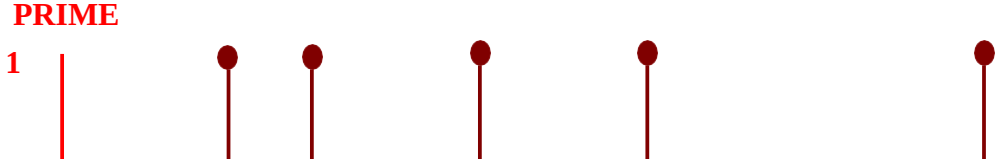


Fig Graphic Interpretation of Fuzzy Sets PRIME

- **Graphic Interpretation of Fuzzy Sets UNIVERSALSPACE**

In any application of sets or fuzzy sets theory, all sets are subsets of a fixed set called universal space or universe of discourse denoted by **X**. Universal space **X** as a fuzzy set is a function equal to **1** for all elements.

The fuzzy set **UNIVERSALSPACE** numbers, defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace □ {1, 12, 1}]

The Set **UNIVERSALSPACE** in set **X** is :

UNIVERSALSPACE = FuzzySet {{1, 1}, {2, 1}, {3, 1}, {4, 1}, {5, 1}, {6, 1},
{7, 1}, {8, 1}, {9, 1}, {10, 1}, {11, 1}, {12, 1}}

Therefore **SetUniversal** is represented as

SetUniversal = FuzzySet [{{1,1},{2,1}, {3,1}, {4,1}, {5,1},{6,1}, {7,1},
{8, 1}, {9, 1}, {10, 1}, {11, 1}, {12, 1}} , UniversalSpace □ {1, 12, 1}]

FuzzyPlot [UNIVERSALSPACE, AxesLable □ {"X", " UNIVERSAL SPACE "}]

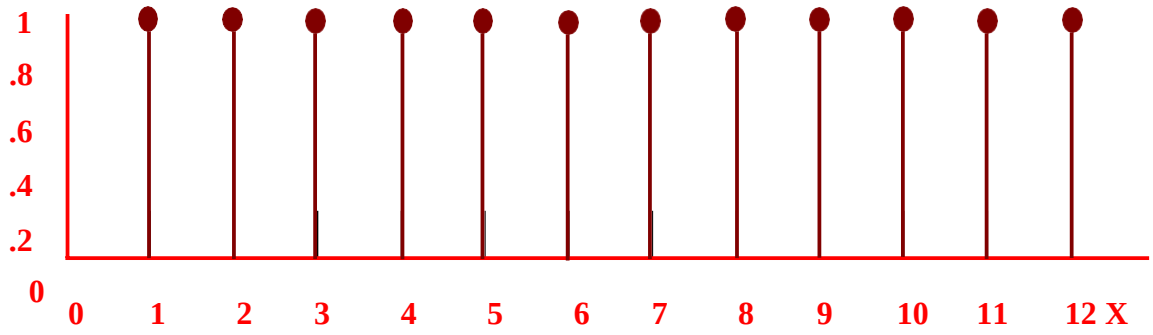


Fig Graphic Interpretation of Fuzzy Set UNIVERSALSPACE

- **Finite and Infinite Universal Space**

Universal sets can be finite or infinite.

Any universal set is finite if it consists of a specific number of different elements, that is, if in counting the different elements of the set, the counting can come to an end, else the set is infinite.

Examples:

1. Let **N** be the universal space of the days of the week.
   N = {Mo, Tu, We, Th, Fr, Sa, Su}. **N** is finite. 2.
   Let **M = {1, 3, 5, 7, 9, ...}.** **M** is infinite.
3. Let **L = {u | u is a lake in a city }.** **L** is finite.

(Although it may be difficult to count the number of lakes in a city, but **L** is still a finite universal set.)

- **Graphic Interpretation of Fuzzy Sets EMPTY**

An empty set is a set that contains only elements with a grade of membership equal to **0**.

Example: Let EMPTY be a set of people, in Minnesota, older than 120. The Empty set is also called the Null set.

The fuzzy set **EMPTY** , defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace □ {1, 12, 1}]

The Set **EMPTY** in set **X** is :

EMPTY = FuzzySet {{1, 0}, {2, 0}, {3, 0}, {4, 0}, {5, 0}, {6, 0}, {7, 0},
{8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}

Therefore **SetEmpty** is represented as

SetEmpty = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0}, {5,0},{6,0}, {7,0},
{8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

FuzzyPlot [EMPTY, AxesLable □ {"X", " UNIVERSAL SPACE "}]

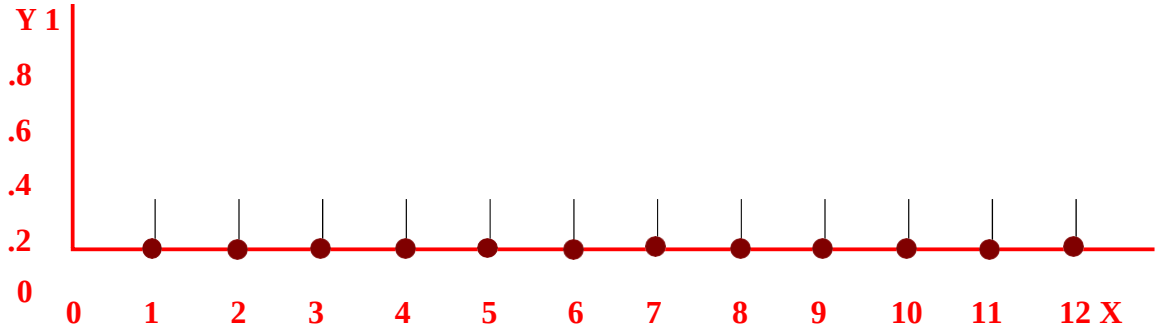


Fig Graphic Interpretation of Fuzzy Set EMPTY

Fuzzy Operations

A fuzzy set operations are the operations on fuzzy sets. The fuzzy set operations are generalization of crisp set operations. Zadeh [1965] formulated the fuzzy set theory in the terms of standard operations: Complement, Union, Intersection, and Difference.

In this section, the graphical interpretation of the following standard fuzzy set terms and the Fuzzy Logic operations are illustrated:

Inclusion :	**FuzzyInclude [VERYSMALL, SMALL]**
Equality :	**FuzzyEQUALITY [SMALL, STILLSMALL]**
Complement :	**FuzzyNOTSMALL = FuzzyCompliment**
[Small] Union :	**FuzzyUNION = [SMALL □ MEDIUM]**
Intersection :	**FUZZYINTERSECTON = [SMALL □**

MEDIUM]

- **Inclusion**

Let **A** and **B** be fuzzy sets defined in the same universal space **X**.

The fuzzy set **A** is included in the fuzzy set **B** if and only if for every **x** in the set **X** we have **A(x) □ B(x)**

Example :

The fuzzy set **UNIVERSALSPACE** numbers, defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace □ {1, 12, 1}]

The fuzzy set B SMALL

The Set **SMALL** in set **X** is :

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},
{7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetSmall** is represented as

SetSmall = FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.4},{6,0.3}, {7,0.2},
{8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

The fuzzy set A VERYSMALL

The Set **VERYSMALL** in set **X** is :

VERYSMALL = FuzzySet {{1, 1 }, {2, 0.8 }, {3, 0.7}, {4, 0.4}, {5, 0.2},
{6, 0.1}, {7, 0 }, {8, 0 }, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetVerySmall** is represented as

SetVerySmall = FuzzySet [{{1,1},{2,0.8}, {3,0.7}, {4,0.4}, {5,0.2},{6,0.1},
{7,0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

The Fuzzy Operation : Inclusion

Include [VERYSMALL, SMALL]

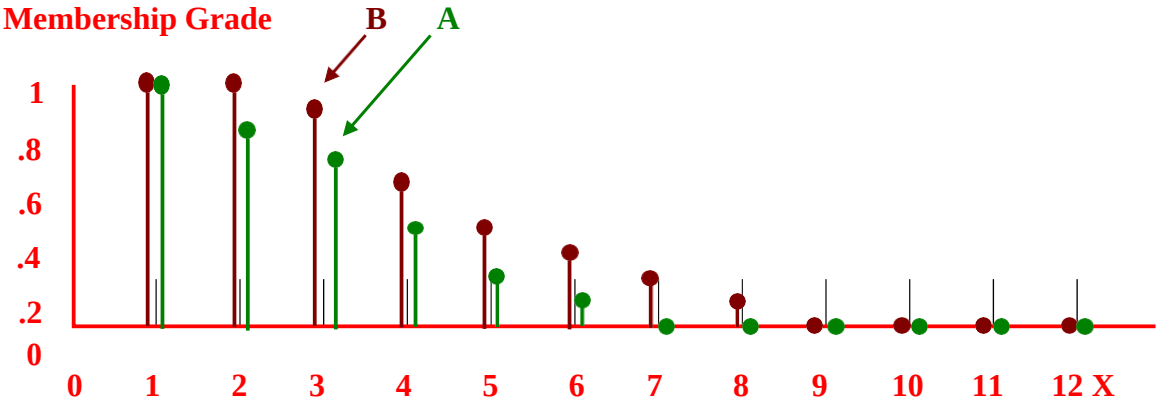


Fig Graphic Interpretation of Fuzzy Inclusion
FuzzyPlot [SMALL, VERYSMALL]

- **Comparability**

Two fuzzy sets **A** and **B** are comparable if the

condition **A □ B or B □ A** holds, ie,

if one of the fuzzy sets is a subset of the other set, they are comparable.

Two fuzzy sets **A** and **B** are incomparable If the

condition **A □ B or B □ A** holds.

Example 1:

Let **A = {{a, 1}, {b, 1},{c, 0}}** **and**

B = {{a, 1}, {b, 1}, {c, 1}}.

Then **A** is comparable to **B**, since **A** is a subset of **B**.

Example 2 :

Let **C = {{a, 1}, {b, 1}, {c, 0.5}} and**

D = {{a, 1}, {b, 0.9}, {c, 0.6}}.

Then **C** and **D** are not comparable since

C is not a subset of **D** and

D is not a subset of **C**.

Property Related to Inclusion :

for all **x** in the set **X**, if **A(x) □ B(x) □ C(x), then accordingly A □ C.**

- **Equality**

Let **A** and **B** be fuzzy sets defined in the same space **X**.

Then **A** and **B** are equal, which is denoted **X = Y**

if and only if for all **x** in the set **X**, **A(x) = B(x).**

Example.

The fuzzy set B SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},

{7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set A STILLSMALL

STILLSMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4},

{6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The Fuzzy Operation : Equality

Equality [SMALL, STILLSMALL]

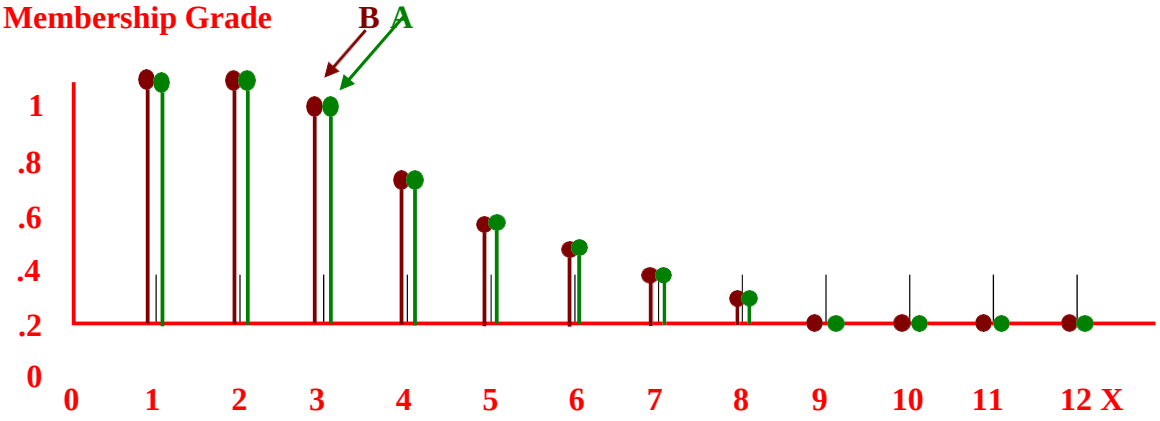


Fig Graphic Interpretation of Fuzzy Equality
FuzzyPlot [SMALL, STILLSMALL]

Note : If equality **A(x) = B(x)** is not satisfied even for one element **x** in the set **X**, then we say that **A** is not equal to **B**.

- **Complement**

Let **A** be a fuzzy set defined in the space **X**.

Then the fuzzy set **B** is a complement of the fuzzy set **A**, if and only if, for all **x** in the set **X**, **B(x) = 1 - A(x).**

The complement of the fuzzy set **A** is often denoted by **A'** or **Ac** or $\bar{A}$

Fuzzy Complement : **Ac(x) = 1 – A(x)**

Example 1.

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},
{7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set Ac NOTSMALL

NOTSMALL = FuzzySet {{1, 0 }, {2, 0 }, {3, 0.1}, {4, 0.4}, {5, 0.6}, {6, 0.7},
{7, 0.8}, {8, 0.9}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The Fuzzy Operation : Compliment

NOTSMALL = Compliment [SMALL]

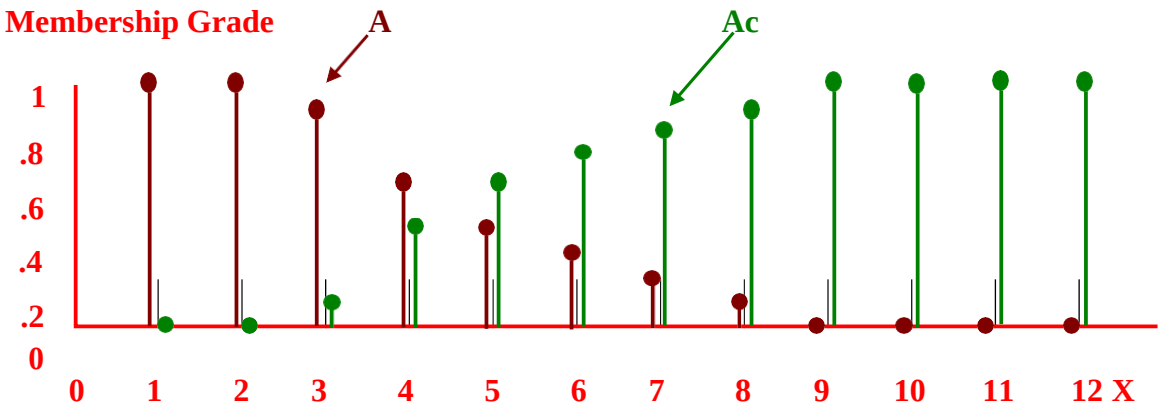


Fig Graphic Interpretation of Fuzzy Compliment
FuzzyPlot [SMALL, NOTSMALL]

Example 2.

The empty set □ and the universal set **X**, as fuzzy sets, are complements of one another.

□' = X , X' = □

The fuzzy set B EMPTY

Empty = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0}, {5, 0}, {6, 0},
{7, 0}, {8, 0}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set A UNIVERSAL

Universal = FuzzySet {{1, 1 }, {2, 1 }, {3, 1}, {4, 1}, {5, 1}, {6, 1},
{7, 1}, {8, 1}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The fuzzy operation : Compliment

EMPTY = Compliment [UNIVERSALSPACE]

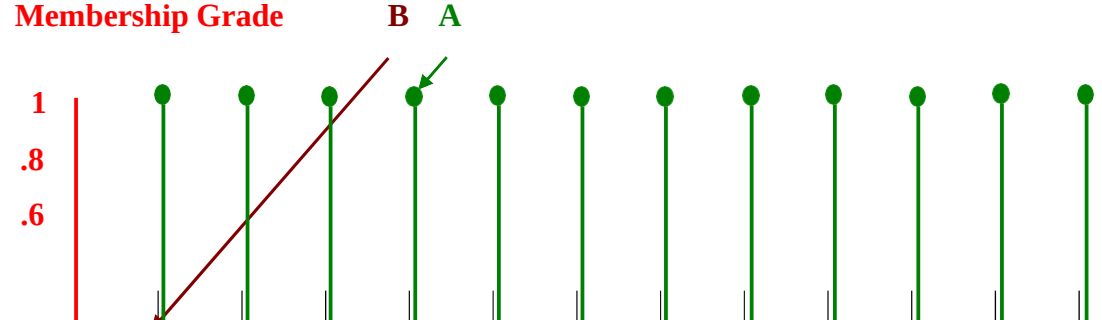


Fig Graphic Interpretation of Fuzzy Compliment
FuzzyPlot [EMPTY, UNIVERSALSPACE]

- **Union**

Let **A** and **B** be fuzzy sets defined in the space **X**.

The union is defined as the smallest fuzzy set that contains both **A** and **B**. The union of **A** and **B** is denoted by **A □ B.**

The following relation must be satisfied for the union operation :

for all x in the set X, (A □ B)(x) = Max (A(x), B(x)).

Fuzzy Union : (A □ B)(x) = max[A(x), B(x)] for all x □ X

Example 1 : Union of Fuzzy **A** and **B**

A(x) = 0.6 and B(x) = 0.4 □ (A □ B)(x) = max [0.6, 0.4] = 0.6

Example 2 : Union of **SMALL** and **MEDIUM**

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},
{7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set B MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8},
{7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12,0}}

The fuzzy operation : Union

FUZZYUNION = [SMALL □ MEDIUM]

SetSmallUNIONMedium = FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5},
{6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}} ,
UniversalSpace □ {1, 12, 1}]

Membership Grade FUZZYUNION = [SMALL □ MEDIUM]

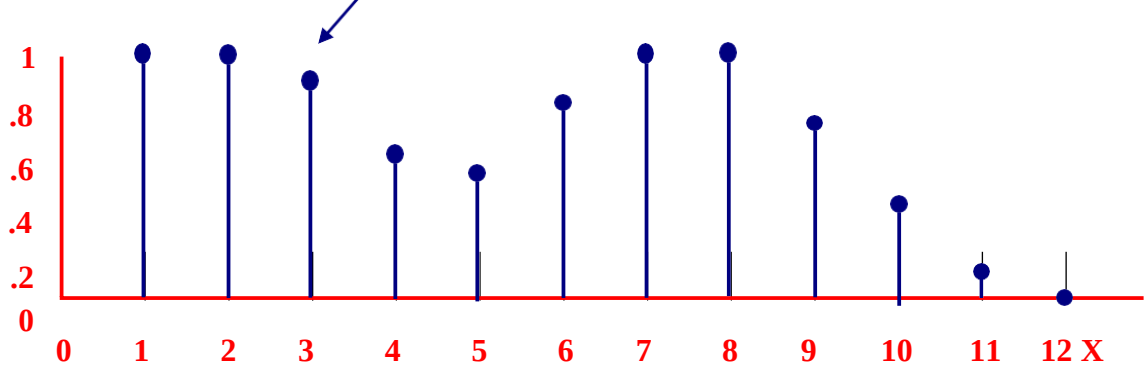


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [UNION]

The notion of the union is closely related to that of the connective "or". Let **A** is a class of "Young" men, **B** is a class of "Bald" men.

If "David is Young" or "David is Bald," then David is associated with the union of **A** and **B.** Implies David is a member of **A □ B**.

- **Intersection**

Let **A** and **B** be fuzzy sets defined in the space **X**. Intersection is defined as the greatest fuzzy set that include both **A** and **B**. Intersection of **A** and **B** is denoted by **A □ B.** The following relation must be satisfied for the intersection operation :

for all x in the set X, (A □ B)(x) = Min (A(x), B(x)).

Fuzzy Intersection : (A □ B)(x) = min [A(x), B(x)] for all x □ X Example 1 :

Intersection of Fuzzy **A** and **B**

A(x) = 0.6 and B(x) = 0.4 □ (A □ B)(x) = min [0.6, 0.4] = 0.4

Example 2 : Union of **SMALL** and **MEDIUM**

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},
{7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set B MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8},
{7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy operation : Intersection

FUZZYINTERSECTION = min [SMALL □ MEDIUM]

SetSmallINTERSECTIONMedium = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0.2},
{5,0.4}, {6,0.3}, {7,0.2}, {8, 0.1}, {9, 0},
{10, 0}, {11, 0}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

Membership Grade **FUZZYINTERSECTON = [SMALL □ MEDIUM]**

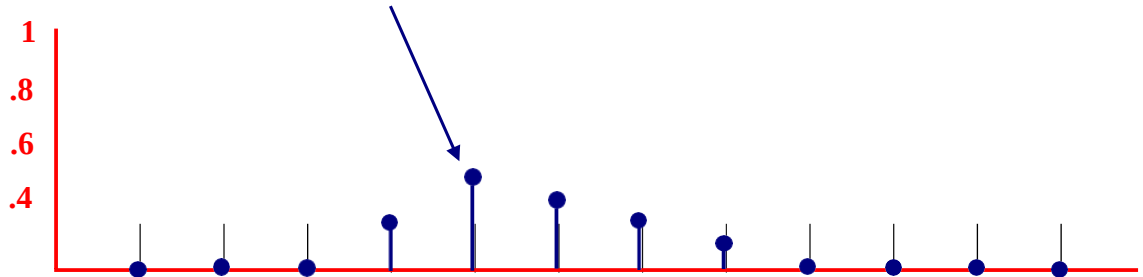


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [INTERSECTION]

- **Difference**

Let **A** and **B** be fuzzy sets defined in the space **X**. The

difference of **A** and **B** is denoted by **A □ B'.**

Fuzzy Difference : **(A - B)(x) = min [A(x), 1- B(x)] for all x □ X**

Example : Difference of **MEDIUM and SMALL**

The fuzzy set A MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8},
{7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy set B SMALL

MEDIUM = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3},
{7, 0.2}, {8, 0.1}, {9, 0.7 }, {10, 0.4 }, {11, 0}, {12, 0}}

Fuzzy Complement : Bc(x) = 1 – B(x)

The fuzzy set Bc NOTSMALL

NOTSMALL = FuzzySet {{1, 0 }, {2, 0 }, {3, 0.1}, {4, 0.4}, {5, 0.6}, {6, 0.7},
{7, 0.8}, {8, 0.9}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The fuzzy operation : Difference by the definition of **Difference**

FUZZYDIFFERENCE = [MEDIUM □ SMALL']

SetMediumDIFFERECESmall = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0.2},
{5,0.5}, {6,0.7}, {7,0.8}, {8, 0.9}, {9, 0.7},
{10, 0.4}, {11, 0.1}, {12, 0}} , UniversalSpace □ {1, 12, 1}]

Membership Grade FUZZYDIFFERENCE = [MEDIUM □ SMALL']

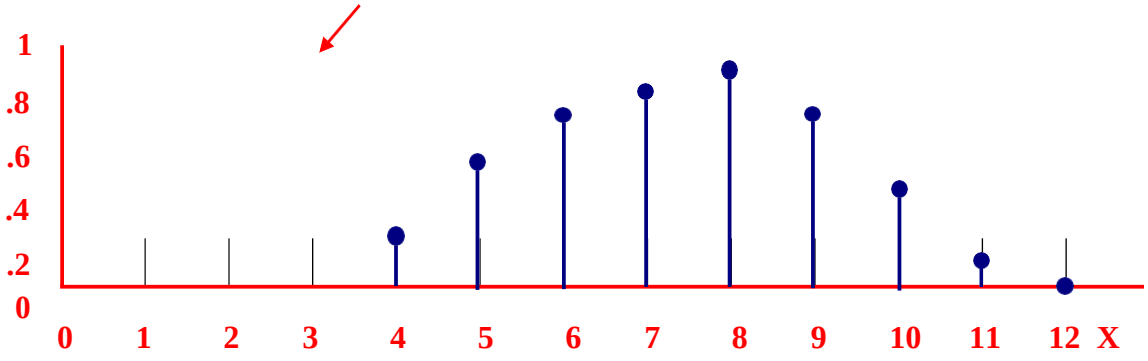


Fig Graphic Interpretation of Fuzzy Union

FuzzyPlot [UNION]

Fuzzy Properties

Properties related to Union, Intersection, Differences are illustrated below.

- **Properties Related to Union**

  The properties related to union are :

  Identity, Idempotence, Commutativity and Associativity.

  - **Identity:**

    A □ □ = A

    input = Equality [SMALL □ EMPTY , SMALL]

    output = True

    A □ X = X

    input = Equality [SMALL □ UnivrsalSpace , UnivrsalSpace]

    output = True

  - **Idempotence :**

    A □ A = A

    input = Equality [SMALL □ SMALL , SMALL]

    output = True

  - **Commutativity :**

    A □ B = B □ A

    input = Equality [SMALL □ MEDIUM, MEDIUM □ SMALL]

    output = True

  [Continued from previous slide]

  - **Associativity:**

    A □ (B□ C) = (A□ B) □ C

    input = Equality [Small □ (Medium □ Big) , (Small □ Medium) □ Big] output = True

    Fuzzy Set Small , Medium , Big

    Small = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0.7 }, {10, 0.4 }, {11, 0}, {12, 0}}

    Medium = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0 }, {10, 0 }, {11, 0.1}, {12, 0}}

Big = FuzzySet [{{1,0}, {2,0}, {3,0}, {4,0}, {5,0}, {6,0.1},
{7,0.2}, {8,0.4}, {9,0.6}, {10,0.8}, {11,1}, {12,1}}]

Calculate Fuzzy relations :

(1) Medium □ Big = FuzzySet [{1,0},{2,0}, {3,0}, {4,0.2}, {5,0.5},
{6,0.8},{7,1}, {8, 1}, {9, 0.6}, {10, 0.8}, {11, 1}, {12, 1}]

(2) Small □ Medium = FuzzySet [{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5},
{6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}]

(3) Small □ (Medium □ Big) = FuzzySet [{1,1},{2,1}, {3,0.9}, {4,0.6},
{5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.8}, {11, 1}, {12, 1}]

(4) (Small □ Medium) □ Big] = FuzzySet [{1,1},{2,1}, {3,0.9}, {4,0.6},
{5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7},{10, 0.8}, {11, 1},{12, 1}]

Fuzzy set (3) and (4) proves Associativity relation

- **Properties Related to Intersection**

  Absorption, Identity, Idempotence, Commutativity, Associativity.

  - **Absorption by Empty Set :**

    A □ □ = □

    input = Equality [Small □ Empty , Empty]

    output = True

  - **Identity :**

    A □ X = A

    input = Equality [Small □ UnivrsalSpace , Small]

    output = True

  - **Idempotence :**

    A □ A = A

    input = Equality [Small □ Small , Small]

    output = True

  - **Commutativity :**

    A □ B = B □ A

    input = Equality [Small □ Big , Big □ Small]

    output = True

  - **Associativity :**

    A □ (B □ C) = (A □ B) □ C

    input = Equality [Small □ (Medium □ Big), (Small □ Medium) □ Big] output = True

- **Additional Properties**

  Related to Intersection and Union

  - **Distributivity:**

    A □ (B □ C) = (A □ B) □ (A □ C)

    input = Equality [Small □ (Medium □ Big) ,
    (Small □ Medium) □ (Small □ Big)]

    output = True

  - **Distributivity:**

    A □ (B □ C) = (A □ B) □ (A □ C)

    input = Equality [Small □ (Medium □ Big) ,
    (Small □ Medium) □ (Small □ Big)]

    output = True

  - **Law of excluded middle :**

    A □ A' = X

    input = Equality [Small □ NotSmall , UnivrsalSpace] output = True

  - **Law of contradiction**

    A □ A' = □

    input = Equality [Small □ NotSmall , EmptySpace] output = True

- **Cartesian Product Of Two Fuzzy Sets**

  - **Cartesian Product of two Crisp Sets**

    Let **A** and **B** be two crisp sets in the universe of discourse **X** and **Y**.. The Cartesian product of **A** and **B** is denoted by **A x B**

    Defined as **A x B = { (a , b) | a □ A , b □ B }**

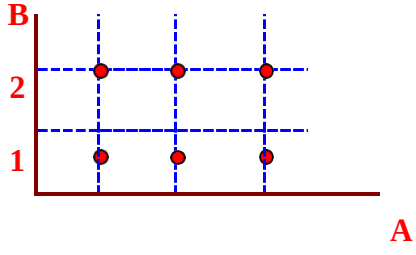
    Note : Generally **A x B ≠ B x A**

    Example :

    Let **A = {a, b, c}** and **B = {1, 2}**

    then **A x B = { (a , 1) , (a , 2) , (b , 1) , (b , 2) , (c , 1) , (c , 2) }**

    Graphic representation of A x B

a **b** **c**

- **Cartesian product of two Fuzzy Sets**

  Let **A** and **B** be two fuzzy sets in the universe of discourse **X** and **Y**. The Cartesian product of **A** and **B** is denoted by **A x B**

  Defined by their membership function $\mu_A(x)$ and $\mu_B(y)$ as

  $$\mu_{A \times B}(x, y) = \min [\mu_A(x), \mu_B(y)] = \mu_A(x) \wedge \mu_B(y)$$

  or $$\mu_{A \times B}(x, y) = \mu_A(x) \cdot \mu_B(y)$$

  for all $x \in X$ and $y \in Y$

  Thus the Cartesian product **A x B** is a fuzzy set of ordered pair **(x , y)** for all $x \in X$ and $y \in Y$, with grade membership of **(x , y)** in **X x Y** given by the above equations.

  In a sense Cartesian product of two Fuzzy sets is a Fuzzy Relation.

2. Fuzzy Relations

Fuzzy Relations describe the degree of association of the elements; Example :
"x is approximately equal to y".

- Fuzzy relations offer the capability to capture the uncertainty and vagueness in relations between sets and elements of a set.
- Fuzzy Relations make the description of a concept possible.
- Fuzzy Relations were introduced to supersede classical crisp relations; It describes the total presence or absence of association of elements.

In this section, first the fuzzy relation is defined and then expressing fuzzy relations in terms of matrices and graphical visualizations. Later the properties of fuzzy relations and operations that can be performed with fuzzy relations are illustrated.

Definition of Fuzzy Relation

Fuzzy relation is a generalization of the definition of fuzzy set from 2-D space to 3-D space.

- **Fuzzy relation definition**

  Consider a Cartesian product

$A \times B = \{ (x , y) \mid x \in A, y \in B \}$

where **A** and **B** are subsets of universal sets **U1** and **U2**.

Fuzzy relation on $A \times B$ is denoted by **R** or $R(x , y)$ is defined as the set

$$R = \{ ((x , y) , \mu_R (x , y)) \mid (x , y) \in A \times B , \mu_R (x , y) \in [0,1] \}$$

where $\mu_R (x , y)$ is a function in two variables called membership function.

- It gives the degree of membership of the ordered pair (x , y) in **R** associating with each pair (x , y) in $A \times B$ a real number in the interval $[0 , 1]$.
- The degree of membership indicates the degree to which **x** is in relation to **y**.

Note :

- Definition of fuzzy relation is a generalization of the definition of fuzzy set from the 2-D space $(x , , \mu_R (x))$ to 3-D space $((x , y) , \mu_R (x , y))$.
- Cartesian product $A \times B$ is a relation by itself between **x** and **y** .
- A fuzzy relation **R** is a sub set of R^3 namely

$$\{ ((x , y) , \mu_R (x , y)) \mid \in A \times B \times [0,1] \in U1 \times U2 \times [0,1] \}$$

- **Example of Fuzzy Relation**

$$R = \{ ((x_1 , y_1) , 0)) , ((x_1 , y_2) , 0.1)) , ((x_1 , y_3) , 0.2)) ,$$
$$((x_2 , y_1) , 0.7)) , ((x_2 , y_2) , 0.2)) , ((x_2 , y_3) , 0.3)) ,$$
$$((x_3 , y_1) , 1)) , ((x_3 , y_2) , 0.6)) , ((x_3 , y_3) , 0.2)) ,$$

The relation can be written in matrix form as

$R \triangleq$

x \ y	y_1	Y_2	Y_3
x_1	0	0.1	0.2
X_2	0.7	0.2	0.3
X_3	1	0.6	0.2

where symbol $\triangleq$ means ' is defined as' and

the values in the matrix are the values of membership function:

$$\mu_R (x_1 , y_1) = 0 \qquad \mu_R (x_1 , y_2) = 0.1 \qquad \mu_R (x_1 , y_3) = 0.2$$
$$\mu_R (x_2 , y_1) = 0.7 \qquad \mu_R (x_2, y_2) = 0.2 \qquad \mu_R (x_2 , y_3) = 0.3$$
$$\mu_R (x_3 , y_1) = 1 \qquad \mu_R (x_3 , y_2) = 0.6 \qquad \mu_R (x_3 , y_3) = 0.2$$

Assuming $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $y_1 = 1$, $y_2 = 2$, $y_3 = 3$,

the relation can be graphically represented by points in 3-D space (X, Y, μ) as :

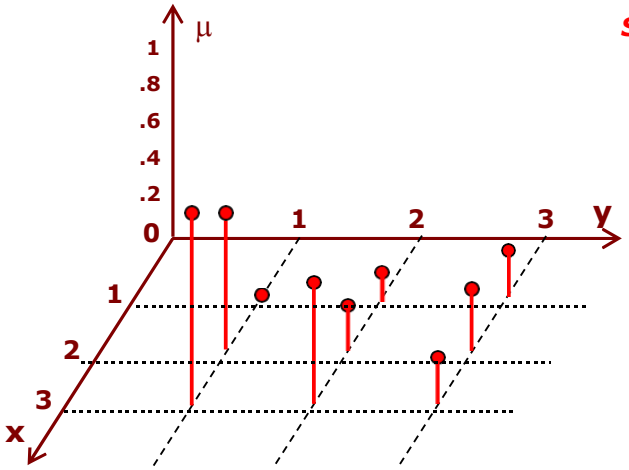


Fig Fuzzy Relation R describing x greater than y

Note : Since the values of the membership function **0.7, 1, 0.6** are in the direction of **x** below the major diagonal **(0, 0.2, 0.2)** in the matrix are grater than those **0.1, 0.2, 0.3** in the direction of **y**, we therefore say that the relation **R** describes **x** is grater than **y**.

Forming Fuzzy Relations

Assume that **V** and **W** are two collections of objects.

A fuzzy relation is characterized in the same way as it is in a fuzzy set.

- The first item is a list containing element and membership grade pairs,

  {{v_1, w_1}, R_{11}}, {{ v_1, w_2}, R_{12}}, ... , {{ v_n, w_m}, R_{nm}}}.

  where **{ v_1, w_1}, { v_1, w_2}, ... , { v_n, w_m}** are the elements of the relation are defined as ordered pairs, and **{ R_{11} , R_{12} , ... , R_{nm}}** are the membership grades of the elements of the relation that range from 0 to 1, inclusive.

- The second item is the universal space; for relations, the universal space consists of a pair of ordered pairs,

  {{ V_{min}, V_{max}, C_1}, { W_{min}, W_{max}, C_2}}.

  where the first pair defines the universal space for the first set and the second pair defines the universal space for the second set.

Example showing how fuzzy relations are represented

Let V = {1, 2, 3} and W = {1, 2, 3, 4}.

A fuzzy relation **R** is, a function defined in the space **V x W**, which takes values from the interval [0, 1] , expressed as **R : V x W □ [0, 1]**

R = FuzzyRelation [{{{1, 1}, 1}, {{1, 2}, 0.2}, {{1, 3}, 0.7}, {{1, 4}, 0}, {{2, 1}, 0.7}, {{2, 2}, 1}, {{2, 3}, 0.4}, {{2, 4}, 0.8}, {{3, 1}, 0}, {{3, 2}, 0.6}, {{3, 3}, 0.3}, {{3, 4}, 0.5}, UniversalSpace □ {{1, 3, 1}, {1, 4, 1}}]

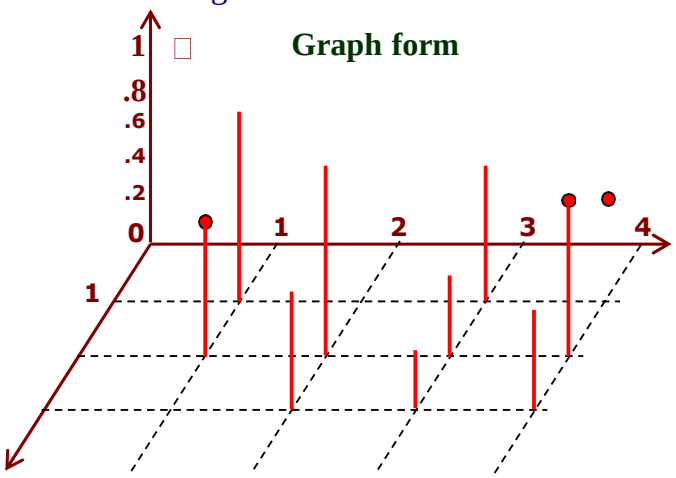
This relation can be represented in the following two forms shown below

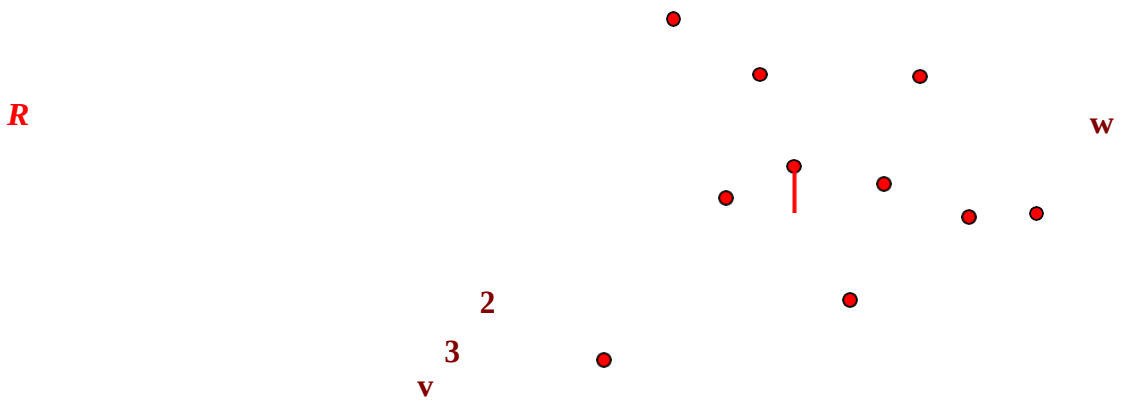
Membership matrix form

$\triangleq$

v \ w	w_1	w_2	w_3	w_4
v_1	1	0.2	0.7	0
v_2	0.7	1	0.4	0.8
v_3	0	0.6	0.3	0.5

Graph form





Vertical lines represent membership grades

Elements of fuzzy relation are ordered pairs $\{v_i , w_j\}$, where v_i is first and w_j is second element. The membership grades of the elements are represented by the heights of the verticallines.

Projections of Fuzzy Relations

Definition : A fuzzy relation on **A x B** is denoted by **R** or **R(x , y)** is defined as the set

$$R = \{ ((x , y) , \mu_R (x , y)) \mid (x , y) \in A \times B , \mu_R (x , y) \in [0,1] \}$$

where $\mu_R (x , y)$ is a function in two variables called membership function. The first, the second and the total projections of fuzzy relations are stated below.

- **First Projection of R :** defined as

$$R^{(1)} = \{(x) , \mu_{R^{(1)}}(x , y))\}$$

$$= \{(x) , \max_Y \mu_R (x , y)) \mid (x , y) \in A \times B \}$$

- **Second Projection of R :** defined as

$$R^{(2)} = \{(y) , \mu_{R^{(2)}}(x , y))\}$$

$$= \{(y) , \max_X \mu R (x , y)) \mid (x , y) \in A \times B \}$$

- **Total Projection of R :** defined as

$$R^{(T)} = \max_X \max_Y \{\mu R (x , y) \mid (x , y) \in A \times B \}$$

Note : In all these three expression

$\max_{Y}$ means **max** with respect to **y** while **x** is considered fixed means **max**

$\max_{X}$ with respect to **x** while **y** is considered fixed

The Total Projection is also known as Global projection

- **Example : Fuzzy Projections**

The Fuzzy Relation **R** together with First, Second and Total Projection of **R** are shown below.

$R \triangleq$

x \ y	y_1	y_2	y_3	y_4	Y_5	$R^{(1)}$
x_1	0.1	0.3	1	0.5	0.3	1
x_2	0.2	0.5	0.7	0.9	0.6	0.9
x_3	0.3	0.6	1	0.8	0.2	1
$R^{(2)}$	0.3	0.6	1	0.9	0.6	1

$= R^{(T)}$

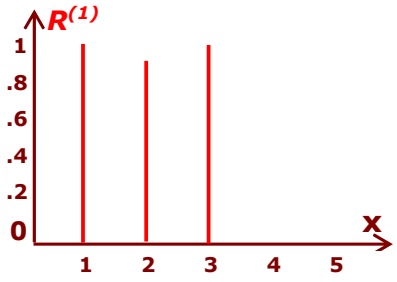
Note :

For $R^{(1)}$ select $\max_{Y}$ means **max** with respect to **y** while **x** is considered fixed means

For $R^{(2)}$ select $\max_{x}$ **max** with respect to **x** while **y** is considered fixed

For $R^{(T)}$ select **max** with respect to $R^{(1)}$ and $R^{(2)}$

The Fuzzy plot of these projections are shown below.



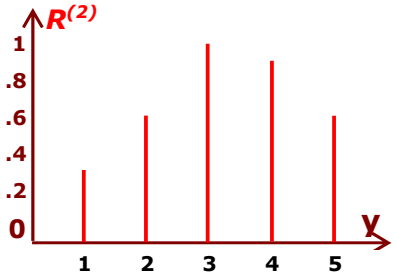


Fig Fuzzy plot of 1st projection $R^{(1)}$ **Fig Fuzzy plot of 2nd projection $R^{(2)}$**

Max-Min and Min-Max Composition

The operation composition combines the fuzzy relations in different variables, say **(x , y)** and **(y , z)** ; **x □ A , y □ B , z □ C .**

Consider the relations :

$R_1(x , y) = \{ ((x , y) , \mu_{R1} (x , y)) \mid (x , y) \in A \times B \}$

$R_2(y , z) = \{ ((y , y) , \mu_{R1} (y , z)) \mid (y , z) \in B \times C \}$

The domain of R_1 is $A \times B$ and the domain of R_2 is $B \times C$

- **Max-Min Composition**

Definition : The Max-Min composition denoted by $R_1 \circ R_2$ with membership function $\mu_{R1 \circ R2}$ defined as

$$R_1 \circ R_2 = \{ ((x , z) , \max_y (\min (\mu_{R1} (x , y) , \mu_{R2} (y , z)))) \},$$
$$(x , z) \in A \times C , y \in B$$

Thus $R_1 \circ R_2$ is relation in the domain $A \times C$

An example of the composition is shown in the next slide.

- **Example : Max-Min Composition**

Consider the relations $R_1(x , y)$ and $R_2(y , z)$ as given below.

$R_1 \triangleq$

x \ y	y_1	y_2	y_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

$R_2 \triangleq$

y \ z	z_1	z_2	z_3
y_1	0.8	0.2	0
y_2	0.2	1	0.6
y_3	0.5	0	0.4

Note : Number of columns in the first table and second table are equal. Compute max-

min composition denoted by $R_1 \circ R_2$:

Step -1 Compute **min** operation (definition in previous slide). Consider row x_1 and column z_1, means the pair (x_1, z_1) for all y_j, $j = 1, 2, 3$, and perform **min** operation

$\min(\mu_{R1}(x_1, y_1), \mu_{R2}(y_1, z_1)) = \min(0.1, 0.8) = 0.1,$

$\min(\mu_{R1}(x_1, y_2), \mu_{R2}(y_2, z_1)) = \min(0.3, 0.2) = 0.2,$

$\min(\mu_{R1}(x_1, y_3), \mu_{R2}(y_3, z_1)) = \min(0, 0.5) = 0,$

Step -2 Compute **max** operation (definition in previousslide).

For $x = x_1$, $z = z_1$, $y = y_j$, $j = 1, 2, 3$,

Calculate the grade membership of the pair (x_1, z_1) as

$\{(x_1, z_1), \max((\min(0.1, 0.8), \min(0.3, 0.2), \min(0, 0.5))\}$

i.e. $\{(x_1, z_1), \max(0.1, 0.2, 0)\}$

i.e. $\{(x_1, z_1), 0.2\}$

Hence the grade membership of the pair (x_1, z_1) is **0.2** .

Similarly, find all the grade membership of the pairs

(x_1, z_2) , (x_1, z_3) , (x_2, z_1) , (x_2, z_2) , (x_2, z_3)

The final result is

$R_1 \circ R_2 =$

x \ z	z_1	z_2	z_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

Note : If tables R_1 and R_2 are considered as matrices, the operation composition resembles the operation multiplication in matrix calculus linking row by columns. After each cell is occupied max-min value (the product is replaced by min, the sum is replaced by max).

- **Example : Min-Max Composition**

The min-max composition is similar to max-min composition with the difference that the roll of max and min are interchanged.

Definition : The max-min composition denoted by $R_1 \square R_2$ with membership function $\mu_{R1 \square R2}$ is defined by

$$R_1 \square R_2 = \{((x, z), \min_y (\max(\mu_{R1}(x, y), \mu_{R2}(y, z))))\}, (x, z) \in A \times C, y \in B$$

Thus $R_1 \square R_2$ is relation in the domain **A x C**

Consider the relations $R_1(x, y)$ and $R_2(y, z)$ as given by the same relation of previous example of max-min composition, that is

$R_1 \triangleq$

x \ y	y_1	y_2	y_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

$R_2 \triangleq$

y \ z	z_1	z_2	z_3
y_1	0.8	0.2	0
y_2	0.2	1	0.6
y_3	0.5	0	0.4

After computation in similar way as done in the case of max-min composition, the final result is

$R_1 \square R_2 =$

x \ z	z_1	z_2	z_3
x_1	0.3	0	0.1
x_2	0.5	0.4	0.4

- **Relation between Max-Min and Min-Max Compositions**

The Max-Min and Min-Max Compositions are related by the formula

$$\overline{R_1} \square \overline{R_2} = \overline{R_1 \square R_2}$$